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Approximate solution of boundary value problems of fourth-order integro-differential equation

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ABSTRACT

In this paper an approximate analytical method for solving a class of two-point boundary value problems for fourth order integro-differential equations is presented. The method is based upon the Laplace transform, perturbation technique and polynomial series. Theoretical considerations are discussed. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique. The results show that the method is of high accuracy and efficient for solving integro-differential equations.

1. Introduction

The theory and application of integral and integro-differential equations is an important subject within applied mathematics and engineering. Integro differential equations are used as mathematical models for many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics [1-4]. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations such as Legendre-collocation spectral approaches [5], homotopy perturbation method [6-8], rationalized Haar functions method [9], Wavelet-Galerkin method [10], differential transform method [11], variational iteration method [12, 13], Lagrange functions [14], Taylor polynomials [15], Chebyshev polynomials [16], sine-cosine wavelets [17], new homotopy perturbation method [18, 19] and Adomian decomposition method [20, 21]. In this paper, we present a simple high accuracy method which is based on combination of the Laplace transform, perturbation technique and polynomial series which have recently been for solving various types of integro-differential equations. The

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outline of this paper is as follows. Section 2 introduces the new approaches for integro-differential equations. Section 3 presents several numerical examples to illustrate the performance and efficiency of the proposed method and Section 4 includes some concluding remarks.

2. Method of Solution

A class of two-point boundary value problems for fourth order integro-differential equations can be considered as follows:

$$y^{(iv)}(x) = f(x) + \gamma y(x) + \int_0^x \left(g(t)y(t) + h(t)F(y(t)) \right) dt,\ \ 0 \le x \le 1, \gamma \in \mathbb{R}, \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = \alpha_0, y(1) = \beta_0, \\ &y'(0) = \alpha_1, y'(1) = \beta_1, \end{aligned} \tag{2}$$

where F is a real nonlinear continuous function, $\gamma, \alpha_0, \beta_0, \alpha_1$ and β_1 are real constants, and f, g and h are given functions and can be approximated by Taylor polynomials.

For solving the equation (1), we construct a following equation

$$y^{(iv)}(x) = u_0(x) - p\left\{ u_0(x) - f(x) - \gamma y(x) - \int_0^x \left(g(t)y(t) + h(t)F(y(t)) \right) dt \right\}, \tag{3}$$

where p is an artificial parameter, $u_0(x) = \sum_{n=0}^{\infty} \gamma_n P_n(x)$ and $\gamma_0, \gamma_1, \gamma_2, \ldots$ are unknown coefficients and $P_0(x), P_1(x), P_2(x), \ldots$ are specific polynomial functions depending on the problem. Obviously, when $p = 1$, from (3) we have original equation (1). By the perturbation technique, assumed that the function $y(x)$ can be expressed by an infinite series, $y(x) = \sum_{n=0}^{\infty} p^n y_n(x)$ and nonlinear term $F(y(x))$ can be decomposed into an infinite series of polynomials given by

$$F(y(x)) = \sum_{n=0}^{\infty} H_n(y_0, y_1, \ldots, y_n) \tag{4}$$

where $H_n(y_0, y_1, \ldots, y_n)$ are called Adomian's polynomials or He's polynomials [22] and are defined by

$$H_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\left[\frac{d^n}{dp^n} F\left(\sum_{k=0}^{n} p^k y_k \right) \right]_{p=0},\ \ n = 0,1,2,\ldots \tag{5}$$

Now let us write the equation (3) in the following form

$$\begin{aligned} y^{(iv)}(x) = \sum_{n=0}^{\infty} \gamma_n P_n(x) - p\Bigg\{ &\sum_{n=0}^{\infty} \gamma_n P_n(x) - \gamma y(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &- \int_0^x \left(\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n y(t) + \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} t^n \sum_{n=0}^{\infty} H_n(y_0, y_1, \ldots, y_n) \right) dt \Bigg\}, \end{aligned} \tag{6}$$

where $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n, g(x)=\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}x^n$ and $h(x)=\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}x^n$ are the Taylor series of $f(x), g(x)$ and $h(x)$ respectively. By applying Laplace transform on both sides of (6), we have

$$\begin{aligned}s^4\mathcal{L}\{y(x)\}-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)=\mathcal{L}\Bigg\{\sum_{n=0}^{\infty}\gamma_nP_n(x)-p\Bigg\{\sum_{n=0}^{\infty}\gamma_nP_n(x)-\gamma y(x)\\ -\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n-\int_0^x\left(\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^ny(t)+\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}t^n\sum_{n=0}^{\infty}H_n(y_0,y_1,...,y_n)\right)dt\Bigg\},\end{aligned}\tag{7}$$

or

$$\begin{aligned}\sum_{n=0}^{\infty}p^ny_n(x)=\mathcal{L}^{-1}\left\{\frac{1}{s^4}\left(\begin{array}{l}s^3\alpha_0+s^2\alpha_1+sA+B+\\ \mathcal{L}\left\{\sum_{n=0}^{\infty}\gamma_nP_n(x)-p\left(\sum_{n=0}^{\infty}\gamma_nP_n(x)-\gamma\sum_{n=0}^{\infty}p^ny_n(x)\right)\right\}\end{array}\right)\right\}\\ +p\mathcal{L}^{-1}\left\{\frac{1}{s^4}\left\{\mathcal{L}\left(\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n+\int_0^x\left(\begin{array}{l}\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^ny(t)\\ +\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}t^n\sum_{n=0}^{\infty}H_n(y_0,y_1,...,y_n)\end{array}\right)dt\right)\right\}\right\},\end{aligned}\tag{8}$$

where $A=y''(0)$ and $B=y'''(0)$ are constants that will be determined later by using the boundary conditions at $x=1$. Comparing coefficients of terms with identical powers of p, leads to

$$\begin{aligned}y_0(x)&=\mathcal{L}^{-1}\left\{\frac{1}{s}\alpha_0+\frac{1}{s^2}\alpha_1+\frac{1}{s^3}A+\frac{1}{s^4}B+\frac{1}{s^4}\sum_{n=0}^{\infty}\gamma_n\mathcal{L}\{P_n(x)\}\right\},\\ y_1(x)&=\mathcal{L}^{-1}\left\{-\frac{1}{s^4}\mathcal{L}\left\{\sum_{n=0}^{\infty}\gamma_nP_n(x)-\gamma y_0(x)-\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\right\}\right\},\\ &\quad-\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x\left(\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^ny_0(t)+H_0(y_0(t))\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}t^n\right)dt\right\}\right\},\\ y_{n+1}(x)&=\mathcal{L}^{-1}\left\{-\frac{1}{s^4}\mathcal{L}\left\{\gamma y_n(x)+\int_0^x\left(\begin{array}{l}\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^ny_n(t)\\ +H_n(y_0,y_1,...,y_n)\sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}t^n\end{array}\right)\right\}\right\}, n=1,2,3,\ldots\end{aligned}\tag{9}$$

Now, let us determine $\gamma_0,\gamma_1,\gamma_2,...$ so that $y_1=0$, then from (9) we have $y_n=0, n=2,3,\ldots$. Setting $p=1$, results in the solution of equation (3) as the following:

$$y(x)=y_0(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}\alpha_0+\frac{1}{s^2}\alpha_1+\frac{1}{s^3}A+\frac{1}{s^4}B+\frac{1}{s^4}\sum_{n=0}^{\infty}\gamma_n\mathcal{L}\{P_n(x)\}\right\}.$$

Therefore, in this method, only the first He's polynomials is calculated, and does not need to solve the differential equation in each iteration.

3. Illustrative Examples

In this section, we apply the new method for solution of linear and nonlinear fourth-order integro-diffrential equation.

Example 1. Consider the linear fourth-order integro-diffrential equation

$$y^{(iv)}(x) = x + (x+3)\mathrm{e}^{x} + y(x) - \int_0^x y(t)\,dt, 0 \le x,t \le 1, \tag{10}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y(1) = 1 + \mathrm{e}, \\ &y'(0) = 1, \;\; y'(1) = 2\mathrm{e}. \end{aligned} \tag{11}$$

which has the exact solution $y(x) = 1 + x\,\mathrm{e}^{x}$. To solve equation (10), by the new method we construct the following equation

$$y^{(iv)}(x) = u_0(x) - p\left\{u_0(x) - x - (x+3)\mathrm{e}^{x} - y(x) + \int_0^x y(t)\,dt\right\}, \tag{12}$$

By applying Laplace transform on both sides of (12), we have

$$\mathcal{L}\left\{y^{(iv)}(x) - u_0(x) + p\left\{u_0(x) - x - (x+3)\mathrm{e}^{x} - y(x) + \int_0^x y(t)\,dt\right\}\right\} = 0$$

Using the differential property of Laplace transform we have

$$\begin{aligned} \mathcal{L}\{y(x)\} = \frac{1}{s^4}\Bigg\{& s^3y(0) + s^2y'(0) + sy''(0) + y'''(0) + \mathcal{L}\Bigg\{\sum_{n=0}^{\infty}\gamma_n P_n(x) \\ & -p\left(\sum_{n=0}^{\infty}\gamma_n P_n(x) - x - (x+3)\mathrm{e}^{x} - y(x) + \int_0^x y(t)\,dt\right)\Bigg\}\Bigg\}, \end{aligned} \tag{13}$$

where $\gamma_0, \gamma_1, \gamma_2, \ldots$ are unknown coefficients, $P_n(x) = x^n$ are specific functions depending on the problem, $y(0) = 1, y'(0) = 1, y''(0) = A, y'''(0) = B$ and $y(x) = \sum_{n=0}^{\infty} p^n y_n(x)$.

By applying inverse Laplace transform on both sides of (13), we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n y_n(x) = \mathcal{L}^{-1}\Bigg\{\frac{1}{s^4}\Bigg\{& s^3 + s^2 + sA + B + \mathcal{L}\Bigg\{\sum_{n=0}^{\infty}\gamma_n P_n(x) \\ & -p\left(\sum_{n=0}^{\infty}\gamma_n P_n(x) - x - (x+3)\mathrm{e}^{x} - \sum_{n=0}^{\infty} p^n y_n(x) + \int_0^x \sum_{n=0}^{\infty} p^n y_n(t)\,dt\right)\Bigg\}, \end{aligned} \tag{14}$$

According to (9) and (14), we have

$$\begin{aligned} y_0(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}A + \frac{1}{s^4}B + \frac{1}{s^4}\sum_{n=0}^{\infty}\gamma_n \mathcal{L}\{P_n(x)\}\right\} \\ &= 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{24}\gamma_0 x^4 + \frac{1}{120}\gamma_1 x^5 + \frac{1}{360}\gamma_2 x^6 + \frac{1}{840}\gamma_3 x^7 + \frac{1}{1680}\gamma_4 x^8 + \cdots, \end{aligned}$$

$$\begin{aligned} y_1(x) &= \mathcal{L}^{-1}\left\{-\frac{1}{s^4}\mathcal{L}\left\{\sum_{n=0}^{\infty}\gamma_n P_n(x) - x - (x+3)\sum_{n=0}^{\infty}\frac{x^n}{n!} - y_0(x) + \int_0^x y_0(t)\,dt\right\}\right\} \\ &= \left(\frac{1}{6} - \frac{1}{24}\gamma_0\right)x^4 + \left(\frac{1}{24} - \frac{1}{120}\gamma_1\right)x^5 + \left(\frac{1}{180} + \frac{1}{720}A - \frac{1}{360}\gamma_2\right)x^6 \\ &+ \left(\frac{1}{840} - \frac{1}{5040}A + \frac{1}{5040}B - \frac{1}{840}\gamma_3\right)x^7 + \cdots, \end{aligned}$$

If we set $y_1(x) = 0$ then we have

$$\gamma_0 = 4, \gamma_1 = 5, \gamma_2 = \frac{1}{2}A + 2, \gamma_3 = 1 - \frac{1}{6}(A - B), \gamma_4 = \frac{11}{24} - \frac{1}{24}B, \ldots$$

We consider the an approximation of $y(x)$ as follows

$$y(x) \approx y_0^*(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \left(\frac{1}{180} + \frac{1}{720}A\right)x^6$$
$$+ \left(\frac{1}{840} - \frac{1}{5040}(A - B)\right)x^7 + \left(\frac{11}{40320} - \frac{1}{40320}B\right)x^8,$$

Imposing the boundary conditions at $x = 1$ on $y_0^*(x)$ we obtain the following linear system

$$\begin{bmatrix} \frac{421}{840} & \frac{961}{5760} \\ \frac{145}{144} & \frac{421}{840} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \mathrm{e} - \frac{49003}{40320} \\ 2\mathrm{e} - \frac{9671}{5040} \end{bmatrix} \tag{15}$$

Solving the system (15) we have

$$A = \frac{34040140}{16905883}\mathrm{e} - \frac{58724761}{16905883} \approx 1.999658187,$$
$$B = -\frac{927360}{16905883}\mathrm{e} - \frac{53258657}{16905883} \approx 3.001193795.$$

Therefore, the approximate solution of equation (10) is

$$y_0^*(x) = 1 + x + 0.99983x^2 + 0.50019x^3 + 0.16667x^4 + 0.04167x^5$$
$$+ 0.00833x^6 + 0.00139x^7 + 0.00020x^8.$$

Some numerical results of these solutions are presented in Table 1 and Figure 1.

Table 1. Numerical values of solutions of Example 1

x_i	$y_{numerical}$	y_{exact}	$\lvert y_{exact} - y_{numerical} \rvert$
0.0	1.000000000	1.000000000	0.0
0.1	1.110515582	1.110517092	1.5E-6
0.2	1.244275308	1.244280552	5.2E-6
0.3	1.404947632	1.404957642	1.0E-5
0.4	1.596715260	1.596729879	1.4E-5
0.5	1.824342723	1.824360636	1.7E-5
0.6	2.093252450	2.093271280	1.8E-5
0.7	2.409610279	2.409626895	1.6E-5
0.8	2.780421517	2.780432742	1.1E-5
0.9	3.213638629	3.213642800	4.1E-6
1.0	3.718281830	3.718281828	2.0E-9

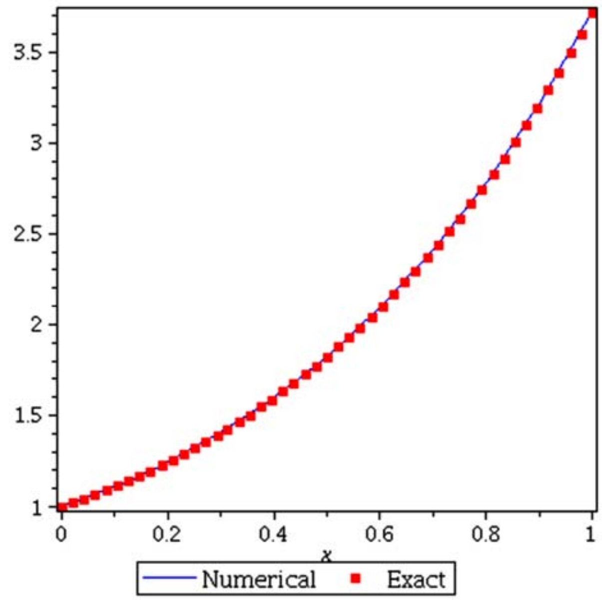

Figure 1. The plot of approximate and exact solutions of example 1

Example 2. Consider the nonlinear fourth-order integro-diffrential equation

$$y^{(iv)}(x) = 1 + \int_0^x \mathrm{e}^{-t}(y(t))^2\,dt, \quad 0 \le x \le 1, \tag{16}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y(1) = \mathrm{e}, \\ &y'(0) = 1, \; y'(1) = \mathrm{e}. \end{aligned} \tag{17}$$

which has the exact solution $y(x) = \mathrm{e}^x$. To solve equation (16), by the new method we construct the following equation

$$y^{(iv)}(x) = u_0(x) - p\left\{u_0(x) - 1 - \int_0^x \mathrm{e}^{-t}(y(t))^2\,dt\right\}, \tag{18}$$

By applying Laplace transform on both sides of (18), we have

$$\mathcal{L}\left\{y^{(iv)}(x) - u_0(x) + p\left\{u_0(x) - 1 - \int_0^x \mathrm{e}^{-t}(y(t))^2\,dt\right\}\right\} = 0$$

Using the differential property of Laplace transform we have

$$\begin{aligned} \mathcal{L}\{y(x)\} = \frac{1}{s^4}\Bigg\{ & s^3 y(0) + s^2 y'(0) + s y''(0) + y'''(0) + \mathcal{L}\Bigg\{\sum_{n=0}^{\infty}\gamma_n P_n(x) \\ & -p\left(\sum_{n=0}^{\infty}\gamma_n P_n(x) - 1 - \int_0^x \mathrm{e}^{-t}(y(t))^2\,dt\right)\Bigg\}\Bigg\} \end{aligned} \tag{19}$$

where $\gamma_0, \gamma_1, \gamma_2, \dots$ are unknown coefficients, $P_n(x) = x^n$ are specific functions depending on the problem, $y(0) = 1,\ y'(0) = 1, y''(0) = A, y'''(0) = B$ and $y(x) = \sum_{n=0}^{\infty} p^n y_n(x)$.

By applying inverse Laplace transform on both sides of (19), we have

$$\sum_{n=0}^{\infty} p^n y_n(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\left\{s^3 + s^2 + sA + B + \mathcal{L}\left\{\sum_{n=0}^{\infty}\gamma_n P_n(x)\right.\right.\right. \\ \left.\left.\left. -p\left(\sum_{n=0}^{\infty}\gamma_n P_n(x) - 1 - \int_0^x \mathrm{e}^{-t}(\sum_{n=0}^{\infty} p^n y_n(x))^2\, dt\right)\right\}\right\}, \tag{20}$$

According to (9) and (20), we have

$$y_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}A + \frac{1}{s^4}B + \frac{1}{s^4}\sum_{n=0}^{\infty}\gamma_n \mathcal{L}\{P_n(x)\}\right\}$$
$$= 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{24}\gamma_0 x^4 + \frac{1}{120}\gamma_1 x^5 + \frac{1}{360}\gamma_2 x^6 + \frac{1}{840}\gamma_3 x^7 + \frac{1}{1680}\gamma_4 x^8 + \cdots,$$
$$y_1(x) = \mathcal{L}^{-1}\left\{-\frac{1}{s^4}\mathcal{L}\left\{\sum_{n=0}^{\infty}\gamma_n P_n(x) - 1 - \int_0^x \mathrm{e}^{-t}(y_0(x))^2\, dt\right\}\right\}$$
$$= \left(\frac{1}{24} - \frac{1}{24}\gamma_0\right)x^4 + \left(\frac{1}{120} - \frac{1}{120}\gamma_1\right)x^5 + \left(\frac{1}{720} - \frac{1}{360}\gamma_2\right)x^6$$
$$+\left(\frac{1}{2520}A - \frac{1}{840}\gamma_3 - \frac{1}{5040}\right)x^7 + \left(\frac{1}{20160}B - \frac{1}{1680}\gamma_4 - \frac{1}{40320}\right)x^8 + \cdots,$$

If we set $y_1(x) = 0$ then we have

$$\gamma_0 = 1, \gamma_1 = 1, \gamma_2 = \frac{1}{2}, \gamma_3 = \frac{1}{3}A - \frac{1}{6}, \gamma_4 = \frac{1}{12}B - \frac{1}{24}, \dots$$

We consider the an approximation of $y(x)$ as follows

$$y(x) \approx y_0^*(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6$$
$$+\left(\frac{1}{2520}A - \frac{1}{5040}\right)x^7 + \left(\frac{1}{20160}B - \frac{1}{40320}\right)x^8,$$

Imposing the boundary conditions at $x = 1$ on $y_0^*(x)$ we obtain the following linear system

$$\begin{bmatrix} \frac{1261}{2520} & \frac{3361}{20160} \\ \frac{361}{360} & \frac{1261}{2520} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \mathrm{e} - \frac{82703}{40320} \\ \mathrm{e} - \frac{1531}{1260} \end{bmatrix} \tag{21}$$

Solving the system (21) we have

$$A = \frac{16952040}{4227721}\mathrm{e} - \frac{83705719}{8455442} \approx 0.999962575,$$
$$B = -\frac{25522560}{4227721}\mathrm{e} + \frac{147211569}{8455442} \approx 1.00013068$$

Therefore, the approximate solution of equation (16) is

$$y_0^*(x) = 1 + x + 0.49998x^2 + 0.16669x^3 + 0.04167x^4 + 0.00833x^5 + 0.00139x^6 + 0.00020x^7 + 0.00002x^8.$$

Some numerical results of these solutions are presented in Table 2 and Figure 2.

Table 2. Numerical values of solutions of Example 2

x_i	$y_{numerical}$	y_{exact}	$\lvert y_{exact} - y_{numerical}\rvert$
0.0	1.000000000	1.000000000	0.0
0.1	1.105170752	1.105170918	1.6E-7
0.2	1.221402186	1.221402728	5.7E-7
0.3	1.349857711	1.349858808	1.0E-6
0.4	1.491823097	1.491824698	1.6E-6
0.5	1.648719310	1.648721271	1.9E-6
0.6	1.822116739	1.822118800	2.0E-6
0.7	2.013750888	2.013752707	1.8E-6
0.8	2.225539701	2.225540928	1.2E-6
0.9	2.459602656	2.459603111	4.5E-7
1.0	2.718281830	2.718281828	2.0E-9

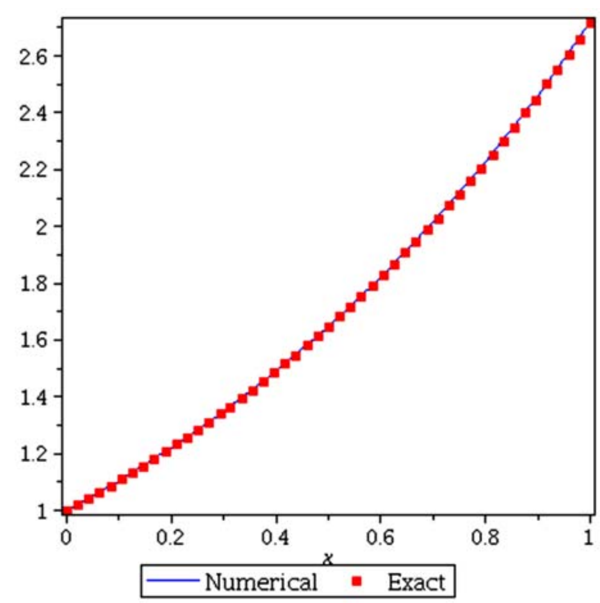

Figure 2. The plot of approximate and exact solutions of example 1

4. Conclusion

In this paper, we have proposed the new efficient method to solve a class of two-point boundary value problems for fourth order integro-differential equations. This approach was based on Laplace transform, perturbation technique and polynomial series. We demonstrate the efficiency and accuracy of the proposed method with some numerical examples. For linear and non-linear fourth order integro-differential we usually derive very good approximations to the solutions. The accuracy of the numerical results indicates that the method is well suited for the solution of such type of problems.

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