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Novel soliton solutions of the generalized (3+1)-dimensional conformable KP and KP–BBM equations

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ABSTRACT

In this study, our main goal is to study the exact traveling wave solutions of some recent nonlinear evolution equations, namely, modified generalized (3+1)-dimensional time-fractional Kadomtsev–Petviashvili (KP) and Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equations of conformable type. We employed a consistent analytical method called the generalized Riccati equation mapping method, along with a conformable derivative to extract the multiple kinks, bi-symmetry soliton, bright and dark soliton, periodic, and singular solutions for suggested equations. The theoretical method is based on the Riccati equation, and a number of empirical solutions have been proposed that do not exist in the literature. Furthermore, as the order of the fractional derivative approaches one, the exact solutions obtained by the current method are reduced to classical solutions. The obtained results show that the present technique is effective, easy to implement, and a strong tool for solving nonlinear fractional partial differential equations and produces a very large number of solutions.

1. Introduction

In recent years, fractional analysis, which has a history as old as classical analysis, has gained rising popularity. Therefore, several phenomena in control theory [1], thermodynamics [2], fluid mechanics [3], signal processing [4], viscoelasticity [5], physics [6], biology [7, 8], dynamical systems [9] and other fields of science have been increasingly interested in modeling through the use of fractional derivatives and integrals. In addition, as they help researchers to adequately address

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and truly comprehend natural phenomena, nonlinear partial fractional differential equations are frequently used in engineering and physical dynamics to model real and actual problems. Since then, numbers of differential equations involving fractional derivatives have been described and studied. After that, calculating exact and approximate solutions of such equations has become very important and a number of analytical and numerical techniques have been researched and established.

The development in symbolic computer programs such as Matlab, Maple, and Mathematica has also led to increasing the number of such methods. As the analytical methods, exp-function method [10], the new extended direct algebraic method [11-13], the new Kudryashov method [14], the unified method [15], the first integral method [16], the generalized exponential rational function method [17], sub-equation method [18, 19], Sardar sub-equation method [20], and simple ansatz method [21], simple equation method and its modification [22-24], and many more [25-34]. Also, since calculating the exact solution of numerous fractional differential equations is complicated, and sometimes impossible, various numerical approaches have been developed. Those procedures include, q-homotopy analysis method [35-37], residual power series method [38-42], the Laplace homotopy perturbation method [43], iterative shehu transform method [44], q-homotopy analysis transform method [45-47], Adomian decomposition method [48-50], reduced differential transform method [46,51], δ-homotopy perturbation transform method [52] can be counted. As mentioned above, many researchers devote mathematical solutions to fractional differential equations. Among them, the generalized Riccati equation mapping method is particularly favored by its power and efficiency in both calculation and analysis. It provides a rich source of solutions, and in particular simple ones. Therefore, this method was preferred in this study.

The purpose of this study is to address the above-mentioned approach with the conformable sense of calculating new exact soliton-type solutions to the generalized (3 + 1)-dimensional time-fractional KP and KP-BBM equations defined respectively as follows [18, 53]:

$$D_t^{\rho}\left(u_x+u_y+u_z\right)+\beta_1 u_{xx}+\beta_2 u_{yy}+\beta_3 u_{zz}+\beta_4\left(u^n u_y\right)_x+\beta_5 u_{xxxy}=0, \tag{1}$$

$$D_t^{\rho}\left(u_x+u_y+u_z\right)+\beta_1 u_{xx}+\beta_2 u_{yy}+\beta_3 u_{zz}+\beta_4\left(u_x u_y\right)_x+\beta_5 u_{xxxy}=0, \tag{2}$$

$$\left(D_t^{\rho}u+\lambda_1 u_x+\lambda_2\left(u^n\right)_x+\lambda_3 D_t^{\rho}u_{xx}\right)_x+\beta_1 u_{yy}+\beta_2 u_{zz}=0, \tag{3}$$

where $\lambda_1,\lambda_2,\lambda_3,\beta_1,\beta_2,\beta_3,\beta_4,$ and β_5 are constants, $\rho,(0<\rho\leq 1)$ is the fractional order and n is a positive real number. Letting $x=y=z$ in Eqs. (1) and (2), then integrate once with respect to $"x"$, after setting the integration constant to zero yields the generalized and potential time-fractional KdV equations respectively as follows:

$$D_t^{\rho}u+au_x+bu^n u_x+cu_{xxx}=0, \tag{4}$$

and

$$D_t^{\rho}u+au_x+bu_x^2+cu_{xxx}=0, \tag{5}$$

where $a=\frac{1}{3}\left(\beta_1+\beta_2+\beta_3\right), b=\frac{\beta_4}{3},$ and $c=\frac{\beta_5}{3}$. In various areas of applied science and engineering, such as water waves, hydrodynamics, quantum field theory and plasma physics, the KdV equations play a vital role. It is important to note that some novel solutions of KP and KP-BBM equations have been studied in [18, 53, 57-61]. However, the solutions obtain in this present study are more general and contains the solutions of the cited references.

The remaining of the manuscript is structured as follows: the definitions and lemmas used in this analysis are provided in section 2 The overview of the generalized Riccati equation mapping method is given in section 3. We apply the technique to the generalized time-fractional equations of KP and KP-BBM in the conformable sense in section 4. Finally, the concluding remarks are dedicated to section 5.

2. Preliminaries

A new fractional calculus definition, called the conforming derivative and its properties, is introduced in this section.

Definition 1. If $u:[0,\infty)\to\mathbb{R}$. Then, we defined the conformable fractional derivative of u of order ρ as

$$D_t^{\rho}u(t)=\lim_{\zeta\to 0}\frac{u(t+\zeta t^{1-\rho})-u(t)}{\zeta},\forall t>0,\rho\in(0,1). \qquad (6)$$

In addition, If u is ρ -differentiable within an interval $(0,\tau)$ where $\tau>0$ and

$$\lim_{t\to 0}u^{(\rho)}(t), \qquad (7)$$

exists, so we specify

$$u^{(\rho)}(0)=\lim_{t\to 0^+}u^{(\rho)}(t). \qquad (8)$$

Lemma 1. [54] Let $\rho\in(0,1]$ and u,v be ρ -differentiable at a point $t>0$. Then

i. $D_t^{\rho}(\theta_1 u+\theta_2 v)=\theta_1 D_t^{\rho}u+\theta_2 D_t^{\rho}v,\forall\,\theta_1,\theta_2\in\mathbb{R}$.

ii. $D_t^{\rho}(t^{ñ})=ñt^{ñ-\rho},\forall\,ñ\in\mathbb{R}$.

iii. $D_t^{\rho}(uv)=uD_t^{\rho}v+vD_t^{\rho}u$.

iv. $D_t^{\rho}\left(\frac{u}{v}\right)=\frac{vD_t^{\rho}u-uD_t^{\rho}v}{v^2}$, provided $v\neq 0$.

v. $D_t^{\rho}(C)=0$, where C is a constant.

vi. $D_t^{\rho}u(t)=t^{1-\rho}\frac{\partial u(t)}{\partial t}$, for the differentiable function u.

3. The Generalized Riccati Equation Mapping (GREM) Method

The primary algorithm for using the GREM method [56] in this section are discussed. Given a nonlinear time-fractional PDE in the form:

$$F\left(u,u_x,u_y,u_z,\cdots,D_t^{\rho}u\right)=0. \qquad (9)$$

The wave transformation $u(x,y,z,t)=u(\eta),\eta=px+qy+rz+c\frac{t^{\rho}}{\rho}$, where c,p,q and r are arbitrary constants to be calculated afterwards is used to rewrite Eq. (9) to nonlinear ODE

$$G\left(u(\eta),u'(\eta),u''(\eta),u'''(\eta),\cdots\right)=0. \qquad (10)$$

Assuming the solutions of Eq. (10) take the form

$$u(\eta) = d_0 + \sum_{i=1}^{N} d_i \varphi^i(\eta),\ d_N \neq 0. \tag{11}$$

Here, $d_i, (i = 0, 1, \cdots, N), c, p, q$ and r are parameters to be calculated later. The integer N can be found by balancing the nonlinear terms and the highest-order derivative in Eq. (10). The function $\varphi(\eta)$ satisfies the generalized Riccati equation:

$$\varphi'(\eta) = R + P\varphi(\eta) + Q\varphi^2(\eta), \tag{12}$$

where R, P and Q are all variable real constants. The families of solutions that justifies Eq. (12) is described as follows:

Family 1. When $\theta = P^2 - 4QR > 0$ and $PQ \neq 0$ or $(QR \neq 0)$:

$$\varphi_1(\eta) = -\frac{1}{2Q}\left(P + \sqrt{\theta}\tanh\left(\frac{\sqrt{\theta}}{2}\eta\right)\right),$$

$$\varphi_2(\eta) = -\frac{1}{2Q}\left(P + \sqrt{\theta}\coth\left(\frac{\sqrt{\theta}}{2}\eta\right)\right),$$

$$\varphi_3(\eta) = -\frac{1}{2Q}\left(P + \sqrt{\theta}\left(\tanh\left(\sqrt{\theta}\eta\right) \pm i\,\mathrm{sech}\left(\sqrt{\theta}\eta\right)\right)\right),$$

$$\varphi_4(\eta) = -\frac{1}{2Q}\left(P + \sqrt{\theta}\left(\coth\left(\sqrt{\theta}\eta\right) \pm \mathrm{csch}\left(\sqrt{\theta}\eta\right)\right)\right),$$

$$\varphi_5(\eta) = -\frac{1}{4Q}\left(2P + \sqrt{\theta}\left(\tanh\left(\frac{\sqrt{\theta}}{4}\eta\right) + \coth\left(\frac{\sqrt{\theta}}{4}\eta\right)\right)\right),$$

$$\varphi_6(\eta) = \frac{1}{2Q}\left(-P + \frac{\sqrt{\left(A^2 + B^2\right)\theta} - A\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)}{A\sinh(\sqrt{\theta}\eta) + B}\right),$$

$$\varphi_7(\eta) = \frac{1}{2Q}\left(-P - \frac{\sqrt{\left(A^2 + B^2\right)\theta} + A\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)}{A\sinh\left(\sqrt{\theta}\eta\right) + B}\right),$$

where two non-zero A and B are real constants.

$$\varphi_8(\eta) = \frac{2R\cosh\left(\frac{\sqrt{\theta}}{2}\eta\right)}{\sqrt{\theta}\sinh\left(\frac{\sqrt{\theta}}{2}\eta\right) - P\cosh\left(\frac{\sqrt{\theta}}{2}\eta\right)},$$

$$\varphi_9(\eta)=\frac{-2R\sinh\left(\frac{\sqrt{\theta}}{2}\eta\right)}{P\sinh\left(\frac{\sqrt{\theta}}{2}\eta\right)-\sqrt{\theta}\cosh\left(\frac{\sqrt{\theta}}{2}\eta\right)},$$

$$\varphi_{11}(\eta)=\frac{2R\sinh\left(\sqrt{\theta}\eta\right)}{\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)-P\sinh\left(\sqrt{\theta}\eta\right)\pm i\sqrt{\theta}},$$

$$\varphi_{10}(\eta)=\frac{2R\cosh\left(\sqrt{\theta}\eta\right)}{\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)-P\cosh\left(\sqrt{\theta}\eta\right)\pm i\sqrt{\theta}},$$

$$\varphi_{12}(\eta)=\frac{4R\sinh\left(\frac{\sqrt{\theta}}{4}\eta\right)\cosh\left(\frac{\sqrt{\theta}}{4}\eta\right)}{2\sqrt{\theta}\cosh^2\left(\frac{\sqrt{\theta}}{4}\eta\right)-2P\sinh\left(\frac{\sqrt{\theta}}{4}\eta\right)\cosh\left(\frac{\sqrt{\theta}}{4}\eta\right)-\sqrt{\theta}}.$$

Family 2. When $\theta=P^2-4QR<0$ and $PQ\neq 0$ (or $QR\neq 0$):

$$\varphi_{13}(\eta)=\frac{1}{2Q}\left(-P+\sqrt{-\theta}\tan\left(\frac{\sqrt{-\theta}}{2}\eta\right)\right),$$

$$\varphi_{14}(\eta)=-\frac{1}{2Q}\left(P+\sqrt{-\theta}\cot\left(\frac{\sqrt{-\theta}}{2}\eta\right)\right),$$

$$\varphi_{15}(\eta)=\frac{1}{2Q}\left(-P+\sqrt{-\theta}\left(\tan\left(\sqrt{-\theta}\eta\right)\pm\sec\left(\sqrt{-\theta}\eta\right)\right)\right),$$

$$\varphi_{16}(\eta)=-\frac{1}{2Q}\left(P+\sqrt{-\theta}\left(\cot\left(\sqrt{-\theta}\eta\right)\pm\csc\left(\sqrt{-\theta}\eta\right)\right)\right),$$

$$\varphi_{17}(\eta)=\frac{1}{4Q}\left(-2P+\sqrt{-\theta}\left(\tan\left(\frac{\sqrt{-\theta}}{4}\eta\right)-\cot\left(\frac{\sqrt{-\theta}}{4}\eta\right)\right)\right),$$

$$\varphi_{18}(\eta)=\frac{1}{2Q}\left(-P+\frac{\pm\sqrt{\theta\left(B^2-A^2\right)}\pm A\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)}{A\sin\sqrt{-\theta}+B}\right),$$

where two non-zero real constants A and B satisfy $A^2-B^2>0$.

$$\varphi_{19}(\eta)=-\frac{2R\cos\left(\frac{\sqrt{-\theta}}{2}\eta\right)}{\sqrt{-\theta}\sin\left(\frac{\sqrt{-\theta}}{2}\eta\right)+P\cos\left(\frac{\sqrt{-\theta}}{2}\eta\right)},$$

$$\varphi_{20}(\eta)=\frac{2R\sin\left(\frac{\sqrt{-\theta}}{2}\eta\right)}{\sqrt{-\theta}\cos\left(\frac{\sqrt{-\theta}}{2}\eta\right)-P\sin\left(\frac{\sqrt{-\theta}}{2}\eta\right)},$$

$$\varphi_{21}(\eta)=-\frac{2R\cos\left(\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)+P\cos\left(\sqrt{-\theta}\eta\right)\pm\sqrt{-\theta}},$$

$$\varphi_{22}(\eta)=-\frac{2R\sin\left(\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+P\sin\left(\sqrt{-\theta}\eta\right)\pm\sqrt{-\theta}},$$

$$\varphi_{23}(\eta)=\frac{4R\sin\left(\frac{\sqrt{-\theta}}{4}\eta\right)\cos\left(\frac{\sqrt{-\theta}}{4}\eta\right)}{2\sqrt{-\theta}\cos^2\left(\frac{\sqrt{-\theta}}{4}\eta\right)-2P\sin\left(\frac{\sqrt{-\theta}}{4}\eta\right)\cos\left(\frac{\sqrt{-\theta}}{4}\eta\right)-\sqrt{-\theta}}.$$

Family 3. When $R=0$ and $PQ\neq 0$:

$$\varphi_{24}(\eta)=\frac{-PA}{Q\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)},$$

$$\varphi_{25}(\eta)=-\frac{P\left(\cosh(P\eta)-\sinh(P\eta)\right)}{Q\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)}.$$

Family 4. When $Q\neq 0$ and $P=R=0$:

$$\varphi_{26}(\eta)=-\frac{1}{Q\eta+\eta_0},$$

where η_0 is an arbitrary constant. Inserting Eq. (11) into Eq. (10), take note of Eq. (12). After setting all coefficients of $\varphi^i(\eta)$ to zero, we achieve some algebraic equations in $c, d_i\, p, q$ and r. Solving these algebraic equations and putting into Eq. (11), also, using the above defined families of solutions, we finally obtain an explicit solutions of Eq. (9).

4. Applications of GREM method to Generalized KP and KP-BMM Equations

Here, we apply GREM method defined in Section 3 and the conformable definition and properties in Section 2 to the generalized (3 + 1)-dimensional time-fractional KP and KP-BBM equations.

Example 1. Given the generalized (3+1)-dimensional time-fractional KP equation as

$$D_t^{\rho}\left(u_x+u_y+u_z\right)+\beta_1 u_{xx}+\beta_2 u_{yy}+\beta_3 u_{zz}+\beta_4\left(u^n u_y\right)_x+\beta_5 u_{xxxy}=0,\quad 0<\rho\leq 1. \qquad (13)$$

We apply wave transform of the form $u(x,y,z,t)=u(\eta), \eta=px+qy+rz+c\frac{t^{\rho}}{\rho}$ to reduce the above equation into a nonlinear ODE:

$$c\left(pu' + qu' + ru'\right)' + \left(p^2\beta_1 + q^2\beta_2 + r^2\beta_3\right)u'' + pq\beta_4\left(u^n u'\right)' + p^3 q\beta_5 u'''' = 0. \quad (14)$$

By twice integrating Eq. (14) and fixing the integration constant to zero, we obtain

$$\left(pc + qc + rc + \beta_1 p^2 + \beta_2 q^2 + \beta_3 r^2\right)u + \frac{pq\beta_4}{n+1}u^{n+1} + \beta_5 p^3 q u'' = 0. \quad (15)$$

Balancing u^{n+1} and u'' in Eq. (15), we get $N = \frac{2}{n}$. For N to be an integer, $n = 1, 2.$

For $n = 1,$ we have

$$u(\eta) = d_0 + d_1\varphi(\eta) + d_2\varphi^2(\eta). \quad (16)$$

Inserting Eqs. (12) and (16) into Eq. (15), gathering the coefficients of $\varphi^i(\eta)$ and equating them to zero. A system of equations is achieved. After solving this system, we get the following cases:

Case I.

$$\xi = \frac{3p^2\beta_5}{\beta_4}, d_0 = -4RQ\xi, d_1 = -4PQ\xi, d_2 = -4Q^2\xi, \theta = P^2 - 4QR, c = -\frac{p^2\beta_1 + q^2\beta_2 + r^2\beta_3 + p^3 q\beta_5\theta}{p+q+r}. \quad (17)$$

The set of solutions using Eq. (16) are defined as follows:

When $\theta > 0$:

$$u_{001} = \theta\xi\, sech^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \quad (18)$$

$$u_{002} = -\theta\xi\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \quad (19)$$

$$u_{003} = \frac{2\theta\xi}{1 \pm i\sinh\left(\sqrt{\theta}\eta\right)}, \quad (20)$$

$$u_{004} = \frac{2A\theta\xi\left(\pm\sqrt{A^2 + B^2}\cosh\left(\sqrt{\theta}\eta\right) - A + B\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(B + A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}, \quad (21)$$

$$u_{005} = \frac{-d_0\theta}{\left(\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right) - P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}, \quad (22)$$

$$u_{006} = \frac{d_0\theta}{\left(P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right) - \sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}, \quad (23)$$

$$u_{007} = \frac{-2d_0\theta\left(1 - i\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(\sqrt{\theta}\left(i + \sinh\left(\sqrt{\theta}\eta\right)\right) - P\cosh\left(\sqrt{\theta}\eta\right)\right)^2}, \quad (24)$$

$$u_{008} = \frac{-2d_0\theta\left(1 + i\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(\sqrt{\theta}\left(-i + \sinh\left(\sqrt{\theta}\eta\right)\right) - P\cosh\left(\sqrt{\theta}\eta\right)\right)^2}. \quad (25)$$

When $\theta<0$:

$$u_{009}=\theta\xi\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{26}$$

$$u_{010}=\theta\xi\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{27}$$

$$u_{011}=\frac{2\theta\xi}{1\pm\sin\left(\sqrt{-\theta}\eta\right)}, \tag{28}$$

$$u_{012}=\frac{2A\theta\xi\left(\pm\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)+A+B\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}, \tag{29}$$

$$u_{013}=\frac{-d_0\theta}{\left(\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}, \tag{30}$$

$$u_{014}=\frac{-d_0\theta}{\left(P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}, \tag{31}$$

$$u_{015}=\frac{-2d_0\theta\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}, \tag{32}$$

$$u_{016}=\frac{-2d_0\theta\left(1-\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\left(1-\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}. \tag{33}$$

When $R=0$:

$$u_{017}=\frac{4P^2\eta_0\xi\left(\cosh(P\eta)-\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)^2}, \tag{34}$$

$$u_{018}=\frac{4P^2\eta_0\xi\left(\cosh(P\eta)+\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)+\sinh(P\eta)\right)^2}. \tag{35}$$

When $P=R=0$:

$$u_{019}=-4\xi\left(\frac{Q}{\eta_0+Q\eta}\right)^2. \tag{36}$$

Case II.

$$\begin{gathered}\xi=\frac{3p^2\beta_5}{\beta_4},\ d_0=-\frac{2\xi}{3}\left(P^2+2QR\right),\ d_1=-4PQ\xi,\ d_2=-4Q^2\xi,\\ \theta=P^2-4QR,\ c=-\frac{p^2\beta_1+q^2\beta_2+r^2\beta_3-p^3q\beta_5\theta}{p+q+r}.\end{gathered} \tag{37}$$

The set of solutions using Eq. (16) are defined as follows:

When $\theta > 0$:

$$u_{020} = \frac{\theta}{3}\xi\left(2-\cosh\left(\sqrt{\theta}\eta\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{38}$$

$$u_{021} = -\frac{\theta}{3}\xi\left(2+\cosh\left(\sqrt{\theta}\eta\right)\right)\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{39}$$

$$u_{022} = \frac{2\theta}{3}\xi\left(\frac{2-i\sinh\left(\sqrt{\theta}\eta\right)}{1+i\sinh\left(\sqrt{\theta}\eta\right)}\right), \tag{40}$$

$$u_{023} = \frac{2\theta}{3}\xi\left(\frac{2+i\sinh\left(\sqrt{\theta}\eta\right)}{1-i\sinh\left(\sqrt{\theta}\eta\right)}\right), \tag{41}$$

$$u_{024} = \frac{\theta}{3}\xi\left(\frac{\pm 6A\sqrt{A^2+B^2}\cosh\left(\sqrt{\theta}\eta\right)-5A^2-2B^2-A^2\cosh\left(2\sqrt{\theta}\eta\right)+2AB\sinh\left(\sqrt{\theta}\eta\right)}{\left(B+A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}\right), \tag{42}$$

$$u_{025} = \frac{2\theta}{3}\xi\left(\frac{4QR-(\theta+2QR)\cosh\left(\sqrt{\theta}\eta\right)+P\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)}{\left(\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right), \tag{43}$$

$$u_{026} = \frac{2\theta}{3}\xi\left(\frac{-4QR-(\theta+2QR)\cosh\left(\sqrt{\theta}\eta\right)+P\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)}{\left(P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right), \tag{44}$$

$$u_{027} = -\frac{4\theta}{3}\xi\left(\frac{\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)\left(4iQR-P\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)+(\theta+2QR)\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(\sqrt{\theta}\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)-P\cosh\left(\sqrt{\theta}\eta\right)\right)^2}\right), \tag{45}$$

$$u_{028} = \frac{4\theta}{3}\xi\left(\frac{\left(i-\sinh\left(\sqrt{\theta}\eta\right)\right)\left(-4iQR-P\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)+(\theta+2QR)\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(\sqrt{\theta}\left(-i+\sinh\left(\sqrt{\theta}\eta\right)\right)-P\cosh\left(\sqrt{\theta}\eta\right)\right)^2}\right). \tag{46}$$

When $\theta < 0$:

$$u_{029} = \frac{\theta}{3}\xi\left(2-\cos\left(\sqrt{-\theta}\eta\right)\right)\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{47}$$

$$u_{030} = \frac{\theta}{3}\xi\left(2+\cos\left(\sqrt{-\theta}\eta\right)\right)\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{48}$$

$$u_{031} = \frac{2\theta}{3}\xi\left(\frac{2+\sin\left(\sqrt{-\theta}\eta\right)}{1-\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{49}$$

$$u_{032}=\frac{2\theta}{3}\xi\left(\frac{2-\sin\left(\sqrt{-\theta}\eta\right)}{1+\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{50}$$

$$u_{033}=\frac{\theta}{3}\xi\left(\frac{\pm 6A\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)+5A^2-2B^2+A^2\cos\left(2\sqrt{-\theta}\eta\right)+2AB\sin\left(\sqrt{-\theta}\eta\right)}{\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{51}$$

$$u_{034}=\frac{2\theta}{3}\xi\left(\frac{4QR-(\theta+2QR)\cos\left(\sqrt{-\theta}\eta\right)-P\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\left(\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{52}$$

$$u_{035}=\frac{2\theta}{3}\xi\left(\frac{4QR+(\theta+2QR)\cos\left(\sqrt{-\theta}\eta\right)+P\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\left(-P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{53}$$

$$u_{036}=\frac{4\theta}{3}\xi\left(\frac{\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)\left(4QR-P\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+(\theta+2QR)\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(\sqrt{-\theta}\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)+P\cos\left(\sqrt{-\theta}\eta\right)\right)^2}\right). \tag{54}$$

When $R=0$:

$$u_{037}=\frac{4\theta}{3}\xi\left(\frac{\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)\left(-4QR-P\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+(\theta+2QR)\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(\sqrt{-\theta}\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)+P\cos\left(\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{55}$$

$$u_{038}=\frac{2P^2}{3}\xi\left(\frac{\left(-\cosh(P\eta)+\sinh(P\eta)\right)\left(-4\eta_0+\left(\eta_0{}^2+1\right)\cosh(P\eta)+\left(\eta_0{}^2-1\right)\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)^2}\right), \tag{56}$$

$$u_{039}=\frac{2P^2}{3}\xi\left(\frac{\left(\cosh(P\eta)+\sinh(P\eta)\right)\left(4\eta_0-\left(\eta_0{}^2+1\right)\cosh(P\eta)+\left(\eta_0{}^2-1\right)\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)+\sinh(P\eta)\right)^2}\right). \tag{57}$$

When $P=R=0$:

$$u_{040}=-4\xi\left(\frac{Q}{\eta_0+Q\eta}\right)^2. \tag{58}$$

For $n=2,$ Eq. (11) reduces to

$$u(\eta)=d_0+d_1\varphi(\eta). \tag{59}$$

Inserting Eq. (59) and (12) into Eq. (15), setting the coefficients of $\varphi^i(\eta)$ to zero. Solving the resulted algebraic equations gives

$$d_0=\pm i\,pP\sqrt{\frac{3\beta_5}{2\beta_4}},d_1=\frac{2d_0Q}{P},\theta=P^2-4QR,c=-\frac{2p^2\beta_1+2q^2\beta_2+2r^2\beta_3-p^3q\beta_5\theta}{2(p+q+r)}. \tag{60}$$

The solutions set using Eq. (59) are:

When $\theta > 0$:

$$u_{041} = -\frac{d_0}{P}\sqrt{\theta}\tanh\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{61}$$

$$u_{042} = -\frac{d_0}{P}\sqrt{\theta}\coth\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{62}$$

$$u_{043} = \frac{d_0}{P}\sqrt{\theta}\left(\pm i\operatorname{sech}(\sqrt{\theta}\eta) - \tanh(\sqrt{\theta}\eta)\right), \tag{63}$$

$$u_{044} = -\frac{d_0}{2P}\sqrt{\theta}\left(\tanh\left(\frac{1}{4}\sqrt{\theta}\eta\right) + \coth\left(\frac{1}{4}\sqrt{\theta}\eta\right)\right), \tag{64}$$

$$u_{045} = \frac{d_0}{P}\sqrt{\theta}\left(\frac{\pm\sqrt{A^2+B^2} - A\cosh(\sqrt{\theta}\eta)}{B + A\sinh(\sqrt{\theta}\eta)}\right), \tag{65}$$

$$u_{046} = \frac{d_0}{P}\left(P + \frac{4QR\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{-P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right) + \sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{66}$$

$$u_{047} = \frac{d_0}{P}\left(P + \frac{4QR\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right) - P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{67}$$

$$u_{048} = \frac{d_0}{P}\left(P + \frac{4QR\cosh(\sqrt{\theta}\eta)}{\sqrt{\theta}\left(\pm i + \sinh(\sqrt{\theta}\eta)\right) - P\cosh(\sqrt{\theta}\eta)}\right), \tag{68}$$

$$u_{049} = \frac{d_0}{P}\sqrt{\theta}\left(\frac{P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right) - \sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right) - P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right). \tag{69}$$

When $\theta < 0$:

$$u_{050} = \frac{d_0}{P}\sqrt{-\theta}\tan\left(\frac{1}{2}\sqrt{-\theta}\,\eta\right), \tag{70}$$

$$u_{051} = \frac{d_0}{P}\sqrt{-\theta}\cot\left(\frac{1}{2}\sqrt{-\theta}\,\eta\right), \tag{71}$$

$$u_{052} = \frac{d_0}{P}\sqrt{-\theta}\left(\sec(\sqrt{-\theta}\eta) \pm \tan(\sqrt{-\theta}\eta)\right) \tag{72}$$

$$u_{053} = \frac{d_0}{P}\sqrt{-\theta}\left(\frac{\pm\sqrt{A^2-B^2} + A\cos\left(\sqrt{-\theta}\eta\right)}{B + A\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{73}$$

$$u_{054} = \frac{d_0}{P}\sqrt{-\theta}\left(P - \frac{4QR\cos\left(\frac{1}{2}\sqrt{-\theta}\,\eta\right)}{P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right) + \sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{74}$$

$$u_{055} = \frac{d_0}{P}\sqrt{-\theta}\left(P + \frac{4QR\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\,\eta\right) - P\sin\left(\frac{1}{2}\sqrt{-\theta}\,\eta\right)}\right), \tag{75}$$

$$u_{056} = \frac{d_0}{P}\sqrt{-\theta}\left(P - \frac{4QR\cos\left(\sqrt{-\theta}\eta\right)}{P\cos\left(\sqrt{-\theta}\eta\right) + \sqrt{-\theta}\left(\pm 1 + \sin\left(\sqrt{-\theta}\eta\right)\right)}\right), \tag{76}$$

$$u_{057} = \frac{d_0}{P}\sqrt{-\theta}\left(\frac{P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right) - \sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right) - P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right). \tag{77}$$

When $R = 0$:

$$u_{058} = d_0\left(\frac{\eta_0 - \cosh\left(P\eta\right) + \sinh\left(P\eta\right)}{\eta_0 + \cosh\left(P\eta\right) - \sinh\left(P\eta\right)}\right), \tag{78}$$

$$u_{059} = d_0\left(\frac{-\eta_0 + \cosh\left(P\eta\right) + \sinh\left(P\eta\right)}{\eta_0 + \cosh\left(P\eta\right) + \sinh\left(P\eta\right)}\right). \tag{79}$$

When $P = R = 0$:

$$u_{060} = \frac{2d_0Q}{P}\left(\frac{1}{\eta_0 + Q\eta}\right). \tag{80}$$

We are now concentrating on the general case where $N = \frac{2}{n}$ and $n \geq 1$. N requires to be an integer to achieve a closed form analytic solution. With the transformation $u = \mathrm{U}^{\frac{1}{n}}$ and integrating twice with the integration constant equals to zero, Eq. (15) reduces to

$$n^2(n+1)\left(pc + qc + rc + \beta_1 p^2 + \beta_2 q^2 + \beta_3 r^2\right)\mathrm{U}^2 + n^2 pq\beta_4 \mathrm{U}^3 + \beta_5 p^3 q\left((1-n^2)(\mathrm{U}')^2 + n(n+1)\mathrm{U}\mathrm{U}''\right) = 0. \tag{81}$$

Balancing U^3 and $\mathrm{U}\mathrm{U}''$, we obtain $N = 2$, thus:

$$\mathrm{U}(\eta) = d_0 + d_1\varphi(\eta) + d_2\varphi^2(\eta). \tag{82}$$

Putting Eqs. (82) and (12) into Eq. (81) and setting the coefficients of $\varphi^i(\eta)$ to zero. Solving the resulting algebraic equations yields

$$d_0 = -\frac{2(n+1)(n+2)p^2\beta_5 QR}{n^2\beta_4}, d_1 = \frac{d_0 P}{R}, d_2 = \frac{d_0 Q}{R}, \theta = P^2 - 4QR, c = -\frac{n^2\left(p^2\beta_1 + q^2\beta_2 + r^2\beta_3\right) + p^3 q\beta_5\theta}{n^2(p+q+r)}. \tag{83}$$

Using Eq. (82), the solutions are:

When $\theta > 0$:

$$u_{061}=\left(-\frac{d_0\theta}{4QR}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^{\frac{1}{n}}, \tag{84}$$

$$u_{062}=\left(\frac{d_0\theta}{4QR}\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^{\frac{1}{n}}, \tag{85}$$

$$u_{063}=\left(\frac{-d_0\theta}{2QR\left(1\pm i\sinh\left(\sqrt{\theta}\eta\right)\right)}\right)^{\frac{1}{n}}, \tag{86}$$

$$u_{064}=\left(\frac{-Ad_0\theta\left(\pm\sqrt{A^2+B^2}\cosh\left(\sqrt{\theta}\eta\right)-A+B\sinh\left(\sqrt{\theta}\eta\right)\right)}{2QR\left(B+A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{87}$$

$$u_{065}=\left(\frac{-d_0\theta}{\left(P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{88}$$

$$u_{066}=\left(\frac{d_0\theta}{\left(P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{89}$$

$$u_{067}=\left(\frac{-2d_0\theta\left(1-i\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(P\cosh\left(\sqrt{\theta}\eta\right)-\sqrt{\theta}\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)\right)^2}\right)^{\frac{1}{n}}, \tag{90}$$

$$u_{068}=\left(\frac{-2d_0\theta\left(1+i\sinh\left(\sqrt{\theta}\eta\right)\right)}{\left(P\cosh\left(\sqrt{\theta}\eta\right)-\sqrt{\theta}\left(-i+\sinh\left(\sqrt{\theta}\eta\right)\right)\right)^2}\right)^{\frac{1}{n}}. \tag{91}$$

When $\theta<0$:

$$u_{069}=\left(-\frac{d_0\theta\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{4QR}\right)^{\frac{1}{n}}, \tag{92}$$

$$u_{070}=\left(-\frac{d_0\theta\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{4QR}\right)^{\frac{1}{n}}, \tag{93}$$

$$u_{071}=\left(\frac{-d_0\theta}{2QR\left(1\pm\sin\left(\sqrt{-\theta}\eta\right)\right)}\right)^{\frac{1}{n}}, \tag{94}$$

$$u_{072}=\left(\frac{-Ad_0\theta\left(\pm\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)+A+B\sin\left(\sqrt{-\theta}\eta\right)\right)}{2QR\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{95}$$

$$u_{073}=\left(\frac{-d_0\theta}{\left(P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{96}$$

$$u_{074}=\left(\frac{-d_0\theta}{\left(\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n}}, \tag{97}$$

$$u_{075}=\left(\frac{-2d_0\theta\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}\right)^{\frac{1}{n}}, \tag{98}$$

$$u_{076}=\left(\frac{2d_0\theta\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}\right)^{\frac{1}{n}}. \tag{99}$$

When $R=0$:

$$u_{077}=\left(\frac{2(n+1)(n+2)p^2\beta_5P^2\eta_0\left(\cosh(P\eta)-\sinh(P\eta)\right)}{n^2\beta_4\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)^2}\right)^{\frac{1}{n}}, \tag{100}$$

$$u_{078}=\left(\frac{2(n+1)(n+2)p^2\beta_5P^2\eta_0\left(\cosh(P\eta)+\sinh(P\eta)\right)}{n^2\beta_4\left(\eta_0+\cosh(P\eta)+\sinh(P\eta)\right)^2}\right)^{\frac{1}{n}}. \tag{101}$$

When $R=P=0$:

$$u_{079}=\left(-\frac{2(n+1)(n+2)p^2\beta_5Q^2}{n^2\beta_4\left(\eta_0+Q\eta\right)^2}\right)^{\frac{1}{n}}, \tag{102}$$

where η_0 is a constant and $\eta=px+qy+rz+c\dfrac{t^\rho}{\rho}$.

Example 2. Consider another generalized (3+1)-dimensional time-fractional KP equation of the form:

$$D_t^{\rho}\left(u_x+u_y+u_z\right)+\beta_1 u_{xx}+\beta_2 u_{yy}+\beta_3 u_{zz}+\beta_4\left(u_x u_y\right)_x+\beta_5 u_{xxxy}=0, \quad 0<\rho\le 1. \qquad (103)$$

Using the wave transform $u(x,y,z,t)=u(\eta), \eta=px+qy+rz+c\frac{t^{\rho}}{\rho}$ to transform Eq. (103) to a nonlinear ODE:

$$c\left(pu'+qu'+ru'\right)'+\beta_1 p^2 u''+\beta_2 q^2 u''+\beta_3 r^2 u''+\beta_4 p^2 q(u'u')'+\beta_5 p^3 q u''''=0. \qquad (104)$$

Integrating the consequent ODE once and fixing the integration constant to zero, we obtain:

$$\left(pc+qc+rc+\beta_1 p^2+\beta_2 q^2+\beta_3 r^2\right)u'+\beta_4 p^2 q\left(u'\right)^2+\beta_5 p^3 q u'''=0. \qquad (105)$$

Balancing $\left(u'\right)^2$ and u''', we get $N=1$ and we have

$$u(\eta)=d_0+d_1\varphi(\eta). \qquad (106)$$

Substituting Eqs. (106) and (12) into Eq. (105) and letting the coefficients of $\varphi^i(\eta)$ to zero. A set of algebraic equations is obtained in d_0, d_1, p, q, r and c. Consequently, we get

$$d_0=d_0, d_1=-\frac{6p\beta_5 Q}{\beta_4}, \theta=P^2-4QR, c=-\frac{p^2\beta_1+q^2\beta_2+r^2\beta_3+p^3 q\beta_5\theta}{p+q+r}. \qquad (107)$$

The solutions of Eq. (103) are:

When $\theta>0$:

$$u_{080}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\sqrt{\theta}\tanh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right), \qquad (108)$$

$$u_{081}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\sqrt{\theta}\coth\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right), \qquad (109)$$

$$u_{082}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\sqrt{\theta}\left(\tanh\left(\sqrt{\theta}\eta\right)\pm i\,\mathrm{sech}\left(\sqrt{\theta}\eta\right)\right)\right), \qquad (110)$$

$$u_{083}=d_0+\frac{3p\beta_5}{2\beta_4}\left(2P+\sqrt{\theta}\left(\coth\left(\frac{1}{4}\sqrt{\theta}\eta\right)+\tanh\left(\frac{1}{4}\sqrt{\theta}\eta\right)\right)\right), \qquad (111)$$

$$u_{084}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\frac{\sqrt{\theta}\left(\pm\sqrt{A^2+B^2}+A\cosh\left(\sqrt{\theta}\eta\right)\right)}{B+A\sinh\left(\sqrt{\theta}\eta\right)}\right), \qquad (112)$$

$$u_{085}=d_0+\frac{12p\beta_5 QR}{\beta_4}\left(\frac{\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \qquad (113)$$

$$u_{086}=d_0+\frac{12p\beta_5QR}{\beta_4}\left(\frac{\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{114}$$

$$u_{087}=d_0+\frac{12p\beta_5QR}{\beta_4}\left(\frac{\cosh\left(\sqrt{\theta}\eta\right)}{P\cosh\left(\sqrt{\theta}\eta\right)-\sqrt{\theta}\left(\pm i+\sinh\left(\sqrt{\theta}\eta\right)\right)}\right). \tag{115}$$

When $\theta<0$:

$$u_{088}=d_0+\frac{3p\beta_5}{\beta_4}\left(P-\sqrt{-\theta}\tan\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right), \tag{116}$$

$$u_{089}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\sqrt{-\theta}\cot\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right), \tag{117}$$

$$u_{090}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\sqrt{-\theta}\left(\pm\sec\left(\sqrt{-\theta}\eta\right)-\tan\left(\sqrt{-\theta}\eta\right)\right)\right), \tag{118}$$

$$u_{091}=d_0+\frac{3p\beta_5}{2\beta_4}\left(2P+\sqrt{-\theta}\left(\cot\left(\frac{1}{4}\sqrt{-\theta}\eta\right)-\tan\left(\frac{1}{4}\sqrt{-\theta}\eta\right)\right)\right), \tag{119}$$

$$u_{092}=d_0+\frac{3p\beta_5}{\beta_4}\left(P+\frac{\sqrt{-\theta}\left(\pm\sqrt{A^2-B^2}+A\cos\left(\sqrt{-\theta}\eta\right)\right)}{B+A\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{120}$$

$$u_{093}=d_0+\frac{12p\beta_5QR}{\beta_4}\left(\frac{\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{121}$$

$$u_{094}=d_0+\frac{12p\beta_5QR}{\beta_4}\left(\frac{\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{122}$$

$$u_{095}=d_0+\frac{12p\beta_5QR}{\beta_4}\left(\frac{\cos\left(\sqrt{-\theta}\eta\right)}{P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(\pm 1+\sin\left(\sqrt{-\theta}\eta\right)\right)}\right). \tag{123}$$

When $R=0$:

$$u_{096}=d_0+\frac{6p\beta_5P\eta_0}{\beta_4\left(\eta_0+\cosh\left(P\eta\right)-\sinh\left(P\eta\right)\right)}, \tag{124}$$

$$u_{097}=d_0+\frac{6p\beta_5P}{\beta_4}\left(\frac{\cosh\left(P\eta\right)+\sinh\left(P\eta\right)}{\eta_0+\cosh\left(P\eta\right)+\sinh\left(P\eta\right)}\right). \tag{125}$$

When $P=R=0$:

$$u_{098}=d_0+\frac{6p\beta_5Q}{\beta_4}\left(\frac{1}{\eta_0+Q\eta}\right). \tag{126}$$

Here, η_0 is a constant and $\eta = px + qy + rz + c\frac{t^{\rho}}{\rho}.$

Example 3. Given a generalized (3+1)-dimensional time-fractional KP–BBM equation defined by

$$\left(D_t^{\rho}u + \lambda_1 u_x + \lambda_2\left(u^n\right)_x + \lambda_3 D_t^{\rho}u_{xx}\right)_x + \beta_1 u_{yy} + \beta_2 u_{zz} = 0, \quad 0 < \rho \le 1. \tag{127}$$

Employing the wave transform $u(x,y,z,t) = u(\eta), \eta = px + qy + rz + c\frac{t^{\rho}}{\rho}$ to reduce Eq. (127) to ODE:

$$p\left(cu' + p\lambda_1 u' + p\lambda_2\left(u^n\right)' + c\,p^2\lambda_3 u'''\right)' + \beta_1 q^2 u'' + \beta_2 r^2 u'' = 0. \tag{128}$$

Integrating the consequent ODE twice and fixing the integration constant to zero, we get

$$\left(pc + \lambda_1 p^2 + \beta_1 q^2 + \beta_2 r^2\right)u + p^2\lambda_2 u^n + c\,p^3\lambda_3 u'' = 0. \tag{129}$$

Balancing u^n and u'', we obtain $N = \frac{2}{n-1}.$ For $n = 2,$ from Eq. (11) one get

$$u(\eta) = d_0 + d_1\varphi(\eta) + d_2\varphi^2(\eta). \tag{130}$$

Proceeding as before, we have the following two cases:

Case I:

$$\mu = \frac{\lambda_3(q^2\beta_1 + r^2\beta_2 + p^2\lambda_1)}{\lambda_2(1 - p^2\lambda_3\theta)}, d_0 = (P^2 + 2QR)\mu, d_1 = 6PQ\mu, d_2 = 6Q^2\mu, c = -\frac{\lambda_2\mu}{\lambda_3 p}, \theta = P^2 - 4QR. \tag{131}$$

Using Eqs. (130) and (131), the solutions of Eq. (127) are:

When $\theta > 0$:

$$u_{099} = -\frac{\theta}{2}\mu\left(2 - \cosh\left(\sqrt{\theta}\eta\right)\right)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{132}$$

$$u_{100} = \frac{\theta}{2}\mu\left(2 + \cosh\left(\sqrt{\theta}\eta\right)\right)\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{133}$$

$$u_{101} = -\theta\mu\left(\frac{2 - i\sinh\left(\sqrt{\theta}\eta\right)}{1 + i\sinh\left(\sqrt{\theta}\eta\right)}\right), \tag{134}$$

$$u_{102} = -\theta\mu\left(\frac{2 + i\sinh\left(\sqrt{\theta}\eta\right)}{1 - i\sinh\left(\sqrt{\theta}\eta\right)}\right), \tag{135}$$

$$u_{103} = \frac{\theta}{2}\mu\left(\frac{\pm 6A\sqrt{A^2 + B^2}\cosh\left(\sqrt{\theta}\eta\right) + 5A^2 + 2B^2 + A^2\cosh\left(2\sqrt{\theta}\eta\right) - 2AB\sinh\left(\sqrt{\theta}\eta\right)}{\left(B + A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}\right), \tag{136}$$

$$u_{104}=\theta\mu\left(\frac{-4QR+(\theta+2QR)\cosh\left(\sqrt{\theta}\eta\right)-P\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)}{\left(P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right), \tag{137}$$

$$u_{105}=\theta\mu\left(\frac{4QR+(\theta+2QR)\cosh\left(\sqrt{\theta}\eta\right)-P\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)}{\left(\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right), \tag{138}$$

$$u_{106}=2i\theta\mu\left(\frac{4iQR-P\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)+(\theta+2QR)\sinh\left(\sqrt{\theta}\eta\right)}{\left((P+i\sqrt{\theta})\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-(iP+\sqrt{\theta})\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right), \tag{139}$$

$$u_{107}=-2i\theta\mu\left(\frac{4iQR+P\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)-(\theta+2QR)\sinh\left(\sqrt{\theta}\eta\right)}{\left((P-i\sqrt{\theta})\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)+(iP-\sqrt{\theta})\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right). \tag{140}$$

When $\theta<0$:

$$u_{108}=-\frac{\theta}{2}\mu\left(2-\cos\left(\sqrt{-\theta}\eta\right)\right)\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{141}$$

$$u_{109}=-\frac{\theta}{2}\mu\left(2+\cos\left(\sqrt{-\theta}\eta\right)\right)\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{142}$$

$$u_{110}=-\theta\mu\left(\frac{2+\sin\left(\sqrt{-\theta}\eta\right)}{1-\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{143}$$

$$u_{111}=\theta\mu\left(\frac{-2+\sin\left(\sqrt{-\theta}\eta\right)}{1+\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{144}$$

$$u_{112}=\frac{\theta}{2}\mu\left(\frac{\pm 6A\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)-5A^2+2B^2-A^2\cos\left(2\sqrt{-\theta}\eta\right)-2AB\sin\left(\sqrt{-\theta}\eta\right)}{\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{145}$$

$$u_{113}=-\theta\mu\left(\frac{4QR-(\theta+2QR)\cos\left(\sqrt{-\theta}\eta\right)-P\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\left(P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{146}$$

$$u_{114}=-\theta\mu\left(\frac{4QR+(\theta+2QR)\cos\left(\sqrt{-\theta}\eta\right)+P\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\left(\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{147}$$

$$u_{115}=2\theta\mu\left(\frac{4QR-P\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+(\theta+2QR)\sin\left(\sqrt{-\theta}\eta\right)}{\left(\left(P+\sqrt{-\theta}\right)\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\left(-P+\sqrt{-\theta}\right)\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{148}$$

$$u_{116}=-2\theta\mu\left(\frac{4QR+P\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)-(\theta+2QR)\sin\left(\sqrt{-\theta}\eta\right)}{\left(\left(P-\sqrt{-\theta}\right)\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\left(P+\sqrt{-\theta}\right)\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right). \tag{149}$$

When $R=0$:

$$u_{117}=-P^2\mu\left(\frac{\left(\cosh(P\eta)-\sinh(P\eta)\right)\left(4\eta_0-\left(1+\eta_0{}^2\right)\cosh(P\eta)+\left(1-\eta_0{}^2\right)\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)^2}\right), \tag{150}$$

$$u_{118}=-P^2\mu\left(\frac{\left(\cosh(P\eta)+\sinh(P\eta)\right)\left(4\eta_0-\left(1+\eta_0{}^2\right)\cosh(P\eta)-\left(1-\eta_0{}^2\right)\sinh(P\eta)\right)}{\left(\eta_0+\cosh(P\eta)+\sinh(P\eta)\right)^2}\right). \tag{151}$$

When $P=R=0$:

$$u_{119}=6\mu\left(\frac{Q}{\eta_0+Q\eta}\right)^2. \tag{152}$$

Case II:

$$\mu=\frac{\lambda_3(q^2\beta_1+r^2\beta_2+p^2\lambda_1)}{\lambda_2(1+p^2\lambda_3\theta)}, d_0=6QR\mu, d_1=6PQ\mu, d_2=6Q^2\mu, c=-\frac{\lambda_2}{\lambda_3 p}\mu, \theta=P^2-4QR. \tag{153}$$

Following the same procedure, the solutions are:

When $\theta>0$:

$$u_{120}=-\frac{3\theta}{2}\mu\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{154}$$

$$u_{121}=\frac{3\theta}{2}\mu\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{155}$$

$$u_{122}=-3\theta\mu\frac{1}{\left(1\pm i\sinh(\sqrt{\theta}\eta)\right)}, \tag{156}$$

$$u_{123}=3A\theta\mu\left(\frac{\pm\sqrt{A^2+B^2}\cosh\left(\sqrt{\theta}\eta\right)+A-B\sinh\left(\sqrt{\theta}\eta\right)}{\left(B+A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}\right), \tag{157}$$

$$u_{124}=-d_0\theta\frac{1}{\left(P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}, \tag{158}$$

$$u_{125}=d_0\theta\frac{1}{\left(\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}, \tag{159}$$

$$u_{126}=-2d_0\theta\left(\frac{1-i\sinh\left(\sqrt{\theta}\eta\right)}{\left(P\cosh\left(\sqrt{\theta}\eta\right)-\sqrt{\theta}\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)\right)^2}\right), \tag{160}$$

$$u_{127}=-2d_0\theta\left(\frac{1+i\sinh\left(\sqrt{\theta}\eta\right)}{\left(P\cosh\left(\sqrt{\theta}\eta\right)+\sqrt{\theta}\left(i-\sinh\left(\sqrt{\theta}\eta\right)\right)\right)^2}\right), \tag{161}$$

$$u_{128}=-d_0\theta\frac{-1}{\left(P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}. \tag{162}$$

When $\theta<0$:

$$u_{129}=-\frac{3\theta}{2}\mu\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{163}$$

$$u_{130}=-\frac{3\theta}{2}\mu\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{164}$$

$$u_{131}=3\theta\mu\frac{1}{\pm1+\sin\left(\sqrt{-\theta}\eta\right)}, \tag{165}$$

$$u_{132}=3A\theta\mu\left(\frac{\pm\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)-A-B\sin\left(\sqrt{-\theta}\eta\right)}{\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}\right), \tag{166}$$

$$u_{133}=-d_0\theta\frac{1}{\left(P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}, \tag{167}$$

$$u_{134}=-d_0\theta\frac{1}{\left(\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}, \tag{168}$$

$$u_{135}=-2d_0\theta\left(\frac{1+\sin\left(\sqrt{-\theta}\eta\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}\right), \tag{169}$$

$$u_{136}=-2d_0\theta\left(\frac{1-\sin\left(\sqrt{-\theta}\eta\right)}{\left(P\cos\left(\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\left(1-\sin\left(\sqrt{-\theta}\eta\right)\right)\right)^2}\right). \tag{170}$$

When $R=0$:

$$u_{137}=6P^2\eta_0\mu\left(\frac{\sinh(P\eta)-\cosh(P\eta)}{\left(\eta_0+\cosh(P\eta)-\sinh(P\eta)\right)^2}\right), \tag{171}$$

$$u_{138}=-6P^2\eta_0\mu\left(\frac{\sinh(P\eta)+\cosh(P\eta)}{\left(\eta_0+\cosh(P\eta)+\sinh(P\eta)\right)^2}\right). \tag{172}$$

When $P=R=0$:

$$u_{139}=6\mu\left(\frac{Q}{\eta_0+Q\eta}\right)^2. \tag{173}$$

For $n=3$, from Eq. (11) with $N=1$, we have

$$u(\eta)=d_0+d_1\varphi(\eta). \tag{174}$$

Proceeding as before, we have the following:

$$\theta=P^2-4QR, \nu=\sqrt{\frac{\lambda_3(q^2\beta_1+r^2\beta_2+p^2\lambda_1)}{\lambda_2(2-p^2\lambda_3\theta)}}, d_0=\pm P\nu, d_1=\frac{2d_0Q}{P}, c=-\frac{2\nu^2\lambda_2}{p\lambda_3}. \tag{175}$$

Following the same procedure, the solutions are:

When $\theta>0$:

$$u_{140}=\pm\nu\sqrt{\theta}\tanh\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{176}$$

$$u_{141}=\pm\nu\sqrt{\theta}\coth\left(\frac{1}{2}\sqrt{\theta}\eta\right), \tag{177}$$

$$u_{142}=\pm i\nu\sqrt{\theta}\left(\frac{\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-i\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)+i\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{178}$$

$$u_{143}=\pm i\nu\sqrt{\theta}\left(\frac{\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)+i\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-i\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{179}$$

$$u_{144}=\pm\frac{1}{4}\nu\sqrt{\theta}\left(3+\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)\tanh\left(\frac{1}{4}\sqrt{\theta}\eta\right), \tag{180}$$

$$u_{145}=\nu\sqrt{\theta}\left(\frac{\pm\sqrt{A^2+B^2}\pm A\cosh\left(\sqrt{\theta}\eta\right)}{B+A\sinh\left(\sqrt{\theta}\eta\right)}\right), \tag{181}$$

$$u_{146}=\pm\nu\sqrt{\theta}\left(\frac{P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}{P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)}\right), \tag{182}$$

$$u_{147}=\nu\sqrt{\theta}\left(\frac{P\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)-\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)}{\sqrt{\theta}\left(i+\sinh\left(\sqrt{\theta}\eta\right)\right)-P\cosh\left(\sqrt{\theta}\eta\right)}\right), \tag{183}$$

$$u_{148}=\nu\sqrt{\theta}\left(\frac{P\left(i-\sinh\left(\sqrt{\theta}\eta\right)\right)+\sqrt{\theta}\cosh\left(\sqrt{\theta}\eta\right)}{\sqrt{\theta}\left(-i+\sinh\left(\sqrt{\theta}\eta\right)\right)-P\cosh\left(\sqrt{\theta}\eta\right)}\right), \tag{184}$$

$$u_{149}=\pm\nu\sqrt{\theta}\left(\frac{\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)-P\left(1+\cosh\left(\sqrt{\theta}\eta\right)\right)}{P\sinh\left(\sqrt{\theta}\eta\right)-\sqrt{\theta}\left(1+\cosh\left(\sqrt{\theta}\eta\right)\right)}\right), \tag{185}$$

$$u_{150}=\pm\nu\sqrt{\theta}\left(\frac{\sqrt{\theta}\sinh\left(\sqrt{\theta}\eta\right)+P\left(1-\cosh\left(\sqrt{\theta}\eta\right)\right)}{P\sinh\left(\sqrt{\theta}\eta\right)+\sqrt{\theta}\left(1-\cosh\left(\sqrt{\theta}\eta\right)\right)}\right). \tag{186}$$

When $\theta<0$:

$$u_{151}=\pm\nu\sqrt{-\theta}\tan\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{187}$$

$$u_{152}=\pm\nu\sqrt{-\theta}\cot\left(\frac{1}{2}\sqrt{-\theta}\eta\right), \tag{188}$$

$$u_{153}=\pm\nu\sqrt{-\theta}\left(\frac{\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{189}$$

$$u_{154}=\pm\nu\sqrt{-\theta}\left(\frac{\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{190}$$

$$u_{155}=\pm\nu\sqrt{-\theta}\left(\frac{\sqrt{A^2-B^2}+A\cos\left(\sqrt{-\theta}\eta\right)}{B+A\sin\left(\sqrt{-\theta}\eta\right)}\right), \tag{191}$$

$$u_{156}=\pm\nu\sqrt{-\theta}\left(\frac{\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{192}$$

$$u_{157}=\pm\nu\sqrt{-\theta}\left(\frac{P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)}\right), \tag{193}$$

$$u_{158}=\pm\nu\sqrt{-\theta}\left(\frac{\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+P\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)}{P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(1+\sin\left(\sqrt{-\theta}\eta\right)\right)}\right), \tag{194}$$

$$u_{159} = \pm\nu\sqrt{-\theta}\left(\frac{\sqrt{-\theta}\cos\left(\sqrt{-\theta}\eta\right)+P\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)}{P\cos\left(\sqrt{-\theta}\eta\right)+\sqrt{-\theta}\left(-1+\sin\left(\sqrt{-\theta}\eta\right)\right)}\right), \quad (195)$$

$$u_{160} = \pm\nu\sqrt{-\theta}\left(\frac{P\left(1+\cos\left(\sqrt{-\theta}\eta\right)\right)-\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\left(1+\cos\left(\sqrt{-\theta}\eta\right)\right)-P\sin\left(\sqrt{-\theta}\eta\right)}\right), \quad (196)$$

$$u_{161} = \pm\nu\sqrt{-\theta}\left(\frac{P\left(1-\cos\left(\sqrt{-\theta}\eta\right)\right)+\sqrt{-\theta}\sin\left(\sqrt{-\theta}\eta\right)}{\sqrt{-\theta}\left(1-\cos\left(\sqrt{-\theta}\eta\right)\right)+P\sin\left(\sqrt{-\theta}\eta\right)}\right). \quad (197)$$

When $R=0$:

$$u_{162} = \pm P\nu\left(\frac{\eta_0-\cosh\left(P\eta\right)+\sinh\left(P\eta\right)}{\eta_0+\cosh\left(P\eta\right)-\sinh\left(P\eta\right)}\right), \quad (198)$$

$$u_{163} = \pm P\nu\left(\frac{\eta_0-\cosh\left(P\eta\right)-\sinh\left(P\eta\right)}{\eta_0+\cosh\left(P\eta\right)+\sinh\left(P\eta\right)}\right). \quad (199)$$

When $P=R=0$:

$$u_{164} = \pm\frac{2Q\nu}{\eta_0+Q\eta}, P=R=0. \quad (200)$$

We concentrate on the general case where $N=\frac{2}{n-1}$. To get an analytic solution in closed form, we employ the transformation $u=\mathrm{U}^{\frac{1}{n-1}}$ to reduce Eq. (129) after integrating twice and setting integration constant to zero

$$(n-1)^2\left(pc+\lambda_1 p^2+\beta_1 q^2+\beta_2 r^2\right)\mathrm{U}^2+(n-1)^2p^2\lambda_2\mathrm{U}^3+cp^3\lambda_2\left((2-n)(\mathrm{U}')^2+(n-1)\mathrm{U}\mathrm{U}''\right)=0. \quad (201)$$

Balancing U^3 and $\mathrm{U}\mathrm{U}''$ in Eq. (201), $N=2$ and we have

$$\mathrm{U}(\eta)=d_0+d_1\varphi(\eta)+d_2\varphi^2(\eta). \quad (202)$$

Inserting Eqs. (202) and (12) into Eq. (201), then setting the coe_cients of $\varphi^i(\eta)$ to zero, we obtain

$$\theta=P^2-4QR, \upsilon=\frac{\lambda_3(n+1)(q^2\beta_1+r^2\beta_2+p^2\lambda_1)}{\lambda_2\left((n-1)^2+p^2\lambda_3\theta\right)}, d_0=2QR\upsilon, d_1=2QP\upsilon, d_2=2Q^2\upsilon, c=-\frac{\lambda_2\upsilon(n-1)^2}{p\lambda_3(n+1)}. \quad (203)$$

The solutions are:

When $\theta>0$:

$$u_{165}=\left(-\frac{\upsilon\theta}{2}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^{\frac{1}{n-1}}, \quad (204)$$

$$u_{166}=\left(\frac{\upsilon\theta}{2}\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^{\frac{1}{n-1}}, \quad (205)$$

$$u_{167}=\left(\frac{\upsilon\theta}{-1\pm i\sinh\left(\sqrt{\theta}\eta\right)}\right)^{\frac{1}{n-1}}, \tag{206}$$

$$u_{168}=\left(A\upsilon\theta\left(\frac{\pm\sqrt{A^2+B^2}\cosh\left(\sqrt{\theta}\eta\right)+A-B\sinh\left(\sqrt{\theta}\eta\right)}{\left(B+A\sinh\left(\sqrt{\theta}\eta\right)\right)^2}\right)\right)^{\frac{1}{n-1}}, \tag{207}$$

$$u_{169}=\left(\frac{-d_0\theta}{\left(\sqrt{\theta}\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-P\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{208}$$

$$u_{169}=\left(\frac{d_0\theta}{\left(P\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\sqrt{\theta}\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}} \tag{209}$$

$$u_{170}=\left(\frac{2d_0\theta}{\left(\left(P+\sqrt{-\theta}\right)\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)-\left(iP+\sqrt{\theta}\right)\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{210}$$

$$u_{171}=\left(\frac{2d_0\theta}{\left(\left(P-\sqrt{-\theta}\right)\sinh\left(\frac{1}{2}\sqrt{\theta}\eta\right)+\left(iP-\sqrt{\theta}\right)\cosh\left(\frac{1}{2}\sqrt{\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}. \tag{211}$$

When $\theta<0$:

$$u_{172}=\left(-\frac{\upsilon\theta}{2}\sec^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^{\frac{1}{n-1}}, \tag{212}$$

$$u_{173}=\left(-\frac{\upsilon\theta}{2}\csc^2\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^{\frac{1}{n-1}}, \tag{213}$$

$$u_{174}=\left(\frac{\upsilon\theta}{-1\pm\sin\left(\sqrt{-\theta}\eta\right)}\right)^{\frac{1}{n-1}}, \tag{214}$$

$$u_{175}=\left(\frac{A\theta\upsilon\left(\pm\sqrt{A^2-B^2}\cos\left(\sqrt{-\theta}\eta\right)-A-B\sin\left(\sqrt{-\theta}\eta\right)\right)}{\left(B+A\sin\left(\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{215}$$

$$u_{176}=\left(\frac{-d_0\theta}{\left(\sqrt{-\theta}\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+P\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{216}$$

$$u_{177}=\left(\frac{-d_0\theta}{\left(P\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\sqrt{-\theta}\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{217}$$

$$u_{178}=\left(\frac{-2d_0\theta}{\left(\left(P-\sqrt{-\theta}\right)\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)-\left(P+\sqrt{-\theta}\right)\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{218}$$

$$u_{179}=\left(\frac{-2d_0\theta}{\left(\left(P+\sqrt{-\theta}\right)\sin\left(\frac{1}{2}\sqrt{-\theta}\eta\right)+\left(P-\sqrt{-\theta}\right)\cos\left(\frac{1}{2}\sqrt{-\theta}\eta\right)\right)^2}\right)^{\frac{1}{n-1}}. \tag{219}$$

When $R=0$:

$$u_{180}=\left(\frac{2P^2\eta_0\upsilon\left(-\cosh\left(P\eta\right)+\sinh\left(P\eta\right)\right)}{\left(\eta_0+\cosh\left(P\eta\right)-\sinh\left(P\eta\right)\right)^2}\right)^{\frac{1}{n-1}}, \tag{220}$$

$$u_{181}=\left(-\frac{2P^2\eta_0\upsilon\left(\cosh\left(P\eta\right)+\sinh\left(P\eta\right)\right)}{\left(\eta_0+\cosh\left(P\eta\right)+\sinh\left(P\eta\right)\right)^2}\right)^{\frac{1}{n-1}}. \tag{221}$$

When $P=R=0$:

$$u_{182}=\left(2\upsilon\left(\frac{Q}{\eta_0+Q\eta}\right)^2\right)^{\frac{1}{n-1}}. \tag{222}$$

Here, η_0 is a constant and $\eta=px+qy+rz+c\dfrac{t^\rho}{\rho}$.

5. Conclusion

In this research, the generalized (3+1)-dimensional time-fractional KP and KP–BBM equations are investigated with the aid of conformable definition and properties. We developed many precise and general exact solutions which do not appear in the literature using the generalized Riccati equation mapping method and symbolic computations. These solutions include trigonometric, hyperbolic and rational type which are classified into different structures such as the bell-shaped solitons, the kinkshaped solitons, the periodic and singular solutions. All the solutions obtained are checked with the aid of Mathematica symbolic computation program. Undoubtedly, in describing and understanding certain physical characteristics of the considered models in various scientific fields, these existing solutions may play a prominent role.

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