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Triangular functions method for numerical solution of fractional Mathieu equation

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ABSTRACT

Fractional differential equations (FDEs) have recently attracted much attention. Fractional Mathieu equation is a well-known FDE. Here, a method based on operational matrix of triangular functions for fractional order integration is introduced for the numerical solution of fractional Mathieu equation.This technique is a successful method because of reducing the problem to a system of linear equations. By solving this system, an approximate solution is obtained. Illustrative examples demonstrate accuracy and efficiency of the method.

1. Introduction

Fractional calculus as a significant theoretical branch of mathematical theories have recently attracted much attention and have become an increasingly important topic in the literature of differnet fields of science and engineering. A great deal of research has shown the beneficial application of the fractional calculus in a lot of real-life physical systems such as the nonlinear oscillations of bioengineering [1], signal processing [2], hydrologic [3], Viscoelasticity [4].

Fractional calculus is very important to study fractional differential equations (FDEs). As is well-known, FDEs are obtained by replacing integer order derivatives by fractional ones. These equations are more advantageous than integer order differential equations because they are more accurate in simulating natural physical process and dynamic system. Many researches have led to improve and extend FDEs. Some examples of these works can be found in [5].

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Most FDEs cannot be solved analytically. Therefore, various other numerical techniques have been also proposed to solve this type of equations, including fractional-order Lagrange polynomials [6], fractional-order Taylor method [7], finite difference method [8], shifted Chebyshev polynomials [9], and so on.

One well-known FDE which is widely used in physical phenomena such as electromagnetic, parametric oscillators, the motion of a quantum particle in a periodic potential, etc., is fractional Mathieu equation with the following general form [10].

$$D^{\beta}y(t)+kD^{\alpha}y(t)+[a-2q\cos(2t)]y(t)=f(t), \tag{1}$$

where D^{β} and D^{α} represent Caputo derivative operators and $0<\alpha\leq 1$, $0<\beta\leq 2$, a and q are the characteristic number and parameter, respectively, and $f(t)$ is a known function. In the case that $\alpha=1$ and $\beta=2$, $f(t)=0$, Eq. (1) represents the familiar damped Mathieu equation [11].

In the literature, this equation has been studied numerically by some authors. Saberi Najafi has solved it by generalized differential transform method [12]. Pirmohabbati et al. [11] used block-pulse wavelets to solve Eq. (1).

In 2006, Deb et al. introduced a complementary pair of orthogonal functions called triangular (TF) functions based on block pulse functions and used them to analyze dynamic systems [13]. Then, the TF approximation was successfully used to solve variational problems [14], integral equations [15], Volterra Fredholm integro-differential equations [16], nonlinear constrained optimal control problems [17] and Voltterra-Fredelm integral equations [18]. The aim of this research is to apply triangular functions to approximate Eq. (1).

This paper is organized as follows. In Section 2, some basic definitions of fractional calculus and triangular functions are recalled. In section 3, operational matrix of the fractional integration based on TFs is provided. In Section 4, our numerical method for solving the fractional Mathieu equation is presented. In Section 5, the numerical method is illustrated by some examples. Finally, conclusions are presented in Section 6.

2. Priliminaries

In this section, some basic definitions and properties of fractional integaral, derivatives and triangular functions are recalled.

2.1 Fractional calculus

The Riemman–Liouville fractional integral of order $\alpha\geq 0$ of a function f over $[0,+\infty]$ is defined as follows.

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds=\frac{1}{\Gamma(\alpha)}x^{\alpha-1}*f(t), \tag{2}$$

where Γ is the Gamma function and $x^{\alpha-1}*f(t)$ is the convolution product of $x^{\alpha-1}$ and $f(t)$ [19]. The Riemman–Liouville fractional integral has following properties.

$$I^{\alpha}I^{\beta}f(t)=I^{\alpha+\beta}f(t).$$

$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t).$

The Caputo fractional derivative of order $\alpha \geq 0$ of a function f over $[0, +\infty]$ is given by

$$D^{\alpha} f(t) = \begin{cases} I^{n-\alpha} f^{(n)}(t), & n-1 < \alpha < n \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \tag{3}$$

where $n = \lceil \alpha \rceil$ [20].

For the Caputo derivative, we have

$$D^{\alpha} x^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \beta \geq \lceil \alpha \rceil, \\ 0, & \beta < \lceil \alpha \rceil. \end{cases}.$$

$D^{\alpha} I^{\alpha} f(t) = f(t).$

$$I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0^{+}) \frac{t^i}{i!},, t > 0,, n-1 \leq \alpha < n \ .$$

2.2 Triangular functions

Triangular functions have been introuduced by Deb et al. [13].Two m-sets of Triangular functions are defined over the interval $[0, T)$ as follows.

$$T1_i(x) = \begin{cases} 1 - \frac{x-ih}{h}, & (i-1)h \leq x < ih, \\ 0, & otherwise, \end{cases} \tag{4}$$

$$T2_i(x) = \begin{cases} \frac{x-ih}{h}, & (i-1)h \leq x < ih, \\ 0, & otherwise, \end{cases} \tag{5}$$

where $i = 1, \dots, m$, with a positive integer value for m, and $h = \frac{T}{m}$. TFs, are disjoint, orthogonal, and complete [16].

2.2.1 Vector form

Consider m-set TF vectors as **T1**(t)=$[T1_1(t),...,T1_m(t)]^T$, **T2**(t)=$[T2_1(t),...,T2_m(t)]^T$, in which **T1**(t) and **T2**(t) are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively [21]. The product of these two TFs vectors are obtained as follows [16].

$$\mathbf{T1}(t)\mathbf{T1}^T(t) \simeq \begin{pmatrix} T1_0(t) & 0 & \dots & 0 \\ 0 & T1_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T1_{m-1}(t) \end{pmatrix}$$

$$\mathbf{T2}(t)\mathbf{T2}^T(t) \simeq \begin{pmatrix} T2_0(t) & 0 & \dots & 0 \\ 0 & T2_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T2_{m-1}(t) \end{pmatrix}$$

and

$$\mathbf{T1}(t)\mathbf{T2}^T(t) \simeq \mathbf{0}, \quad \textbf{(6)}$$

$$\mathbf{T2}(t)\mathbf{T1}^T(t) \simeq \mathbf{0}, \quad \textbf{(7)}$$

where $\mathbf{0}$ is the zero $m \times m$ matrix. Also,

$$\int_0^1 \mathbf{T1}(t)\mathbf{T1}^T(t)dt = \int_0^1 \mathbf{T2}(t)\mathbf{T2}^T(t)dt \simeq \frac{h}{3}I \quad \textbf{(8)}$$

$$\int_0^1 \mathbf{T1}(t)\mathbf{T2}^T(t)dt = \int_0^1 \mathbf{T2}(t)\mathbf{T1}^T(t)dt \simeq \frac{h}{6}I \quad \textbf{(9)}$$

where I is $m \times m$ identity matrix. Now, define the $2m$-vector $\mathbf{T}(t)$ as follows.

$$\mathbf{T}(t)=[\mathbf{T1}(t),\mathbf{T2}(t)]^T, \quad \textbf{(10)}$$

Then,

$$\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)dt \simeq D, \quad \textbf{(11)}$$

where D is the following $2m \times 2m$ matrix.

$$D = \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix}$$

2.2.2 Function expansion

In general, a square integrable function f can be expanded into an m-set TF series as:

$$f(x) \simeq \sum_{i=1}^{m} c_i T1_i(t) + \sum_{i=1}^{m} d_i T2_i(t) = \mathbf{C}^T\mathbf{T1}(t)(t) + \mathbf{D}^T\mathbf{T2}(t)(t) = \mathbf{F}^T\mathbf{T}(t)(t) \quad \textbf{(12)}$$

where, $c_i = f((i-1)h)$, $d_i = f(ih)$ for $i = 1, \dots, m$, $\mathbf{C} = [c_1, \dots, c_m]^T$, $\mathbf{D} = [d_1, \dots, d_m]^T$, and $\mathbf{F} = [\mathbf{C}, \mathbf{D}]$ [39].

3. Operational matrix of fractional integration

In this section, the operational matrix of fractional integration based on TFs is presented. The fractional integration of $T(t)$ can be approximated as follows.

$$I^\alpha T(t) = P_\alpha T(t), \quad \textbf{(13)}$$

where P_α is the fractional integration operational matrix of $\mathbf{T}(t)$ as follows.

$$P_\alpha = \begin{pmatrix} P1_\alpha & P2_\alpha \\ P3_\alpha & P4_\alpha \end{pmatrix}, \quad \textbf{(14)}$$

where $P1_\alpha, P2_\alpha, P3_\alpha$ and $P4_\alpha$ are $m \times m$ matrices as follows.

$$P1_{\alpha} = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2} \\ 0 & 0 & 0 & \cdots & \zeta_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{15}$$

$$P2_{\alpha} = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{m} \\ 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_1 \end{pmatrix}, \tag{16}$$

$$P3_{\alpha} = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{17}$$

$$P4_{\alpha} = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m} \\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_1 \end{pmatrix}, \tag{18}$$

in which

$$\zeta_r = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left((\alpha+1)r^{\alpha} - r^{\alpha+1} + (r-1)^{\alpha+1}\right),$$

and

$${}_r = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left(r^{\alpha+1} - (r-1)^{\alpha+1} - (\alpha+1)(r-1)^{\alpha}\right).$$

So, from Eq. (12), the fractional integration of $f(t)$ can be approximated as:

$$I^{\alpha} f(t) \simeq F^{T} P_{\alpha} T(t). \tag{19}$$

4. Discussion

In this section, Tfs are employed to solve the fractional Mathieu equation. Consider Eq. (1) with following initial conditions.

$$y(t_0) = y_0, \quad y'(t_0) = y_1 \tag{20}$$

Assume that

$$D^{\beta} y(t) = F^{T} T(t). \tag{21}$$

According to definitions and Eq. (18), $D^{\alpha} y(t)$ can be written as

$$D^{\alpha} y(t) = F^{T} P_{(\beta-\alpha)} T(t) + y_1. \tag{22}$$

So

$$y(t) = F^T P_\beta T(t) + y_0 + y_1 t. \quad \textbf{(23)}$$

Now, integrating from both sides of Eq. (23) with initial conditions leads to

$$F^T T(t) + kF^T P_{\beta-\alpha} T(t) + ky_1 + (a - 2q\cos(2t))(F^T P_\beta T(t) + y_0 + y_1 t) = f(t). \quad \textbf{(24)}$$

By substituting (21)- (24) into (1) following equation is obtained.

$$F^T T(t) + kF^T P_{\beta-\alpha} T(t) + (a - 2q\cos(2t)F^T P_\beta T(t) = g(t), \quad \textbf{(25)}$$

where $g(t) = f(t) - ky_1 - (a - 2q\cos(2t))(y_0 + y_1 t)$ The matrix form of Eq. (25) is

$$F^T\left(I + kP_{(\beta-\alpha)} + \left(a - 2q\cos(2t)\right)P_{(\beta)}\right)T(t) = g(t). \quad \textbf{(26)}$$

Using collocation points $t_i = \frac{(i-1)}{2m}$ for $i = 1, \ldots, 2m$ in (26), a system of linear equations is obtained. Unknown coefficients vector can be determined by solving the resulted system .

5. Numerical experiments

In order to test the proposed method, consider Eq. (1) with $f(t) = e^{-t}$, $k = 1$, $a = 1$, $q = 0$ and initial values $y(0) = 1$ and $y'(0) = 1$. In this case, the exact solution is $y(t) = e^{-t}$. In table 1, the absolute error of numerical solution with $\alpha = 1$ and $\beta = 2$ is presented. This Table shows the absolute error, the difference between the exact and the approximate solution of the method for different values of t and $m = 10$ and $m = 40$. Comparison of errors shows that with increasing sub-intervals, the accuracy of the method also increases.

Now, by keeping the previous coefficients constant, we change α and β from 2 and 1 to 1.9 and 0.8 first and then 1.8 and 0.7, respectively. The changes are compared in Figure 1. According to this figure, although the values of the α and β derivatives in the equation have changed from integer to fractional, it is clear that more accurate solutions are achieved with increasing number of operating subintervals for the triangular functions.

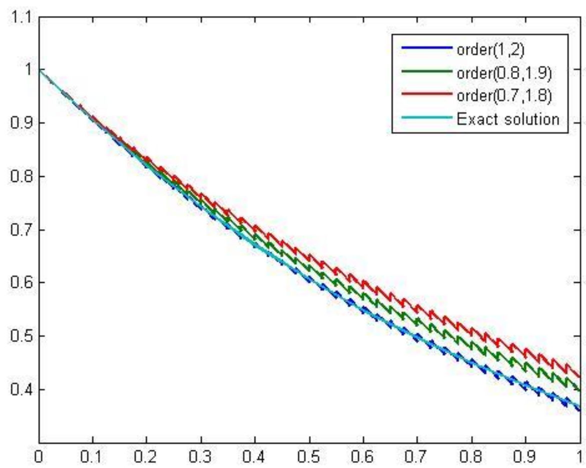

Figure1: Comparison of numerical solutions with exact solution for different values of α and β

Table 1. Absolute error of the proposed method for $m = 10$ and $m = 40$

t	$m = 10$	$m = 40$
0.1	0.0061	0.0013
0.2	0.0101	0.0023
0.3	0.0137	0.0033
0.4	0.0168	0.0041
0.5	0.0194	0.0048
0.6	0.0217	0.0053
0.7	0.0235	0.0058
0.8	0.0249	0.0062
0.9	0.0259	0.0065

6. Conclusions

This paper presents a computational method based on the TFs for solving the fractional Mathieu equation. A system of linear equations is derived by using the operational matrix of the TF for fractional integration. Unknown coefficients can be obtained by solving this system. The numerical results for differnet values of fractional orders have been compared with each other. The results show that this method has a very good accuracy. Because of simplicity of this method, it can be suitable for solving nonlinear equtions such as the Duffing, Bratu, and other similar equations

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