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A maximum entropy approach for the N policy M/G/1 queueing system with a removable server

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ABSTRACT

The maximum entropy principle has grown progressively more pertinent to queueing systems. The principle of maximum entropy presents an impartial framework as a promising method to examine complex queuing processes. In this research, the N policy M/G/1 queueing system with a removable server was analyzed by using the maximum entropy method. We use maximum entropy principle to derive the approximate formulas for the steady-state probability distributions of the queue length. The maximum entropy approach is then used to give a comparative perusal between the system's exact and estimated waiting times. We demonstrate that the maximum entropy approach is efficient enough for practical purpose and is a feasible method for approximating the solution of complex queueing systems.

1. Introduction

Optimization of queuing systems (reducing the length of the queue and reducing the waiting time) is done according to the assumed appropriate cost functions and based on interesting measures such as the expected number of customers in the system, the expected waiting time of customers in the system, the expected period of employment (or unemployment) of servers and.... To obtain these measurements, it is necessary to identify the variables in the model and obtain their distributions. But these distributions have parameters that are usually unknown. so we have to estimate them. Usually, the processes of entry or service or both are random, so the interesting measures for checking the system, that is, values such as waiting time, the number of customers in the system, the duration of the server employment period or unemployment, will also be random variables, and determining the probability distribution of these random variables or the minimum mathematical expectation values They are considered Sometimes it is difficult or impossible to use classical

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methods to find the explicit formulas of these probability distributions. In these cases, we use numerical methods, One of these methods is the maximum entropy method. The term entropy was used for the first time in thermodynamics.

The maximum entropy principle is a probability inference method which has been widely applied in fields such as statistical mechanics and computer performance analysis. Boltzmann (1877) used its probabilistic interpretation in statistical mechanics. Planck (1906) documented the relationship between entropy and probability, and Shannon, in his famous paper in (1948), used this concept to describe the properties of a sequence of symbols and defined it as a criterion for information theory [1]. Shannon's entropy is a measure of the uncertainty of a system, that is, the system with more uncertainty has less reliability. If we focus on its statistical dimension, we will see the relationship between entropy and uncertainty. In fact, uncertainty can be expressed using entropy. Shannon's entropy is used in different branches of statistics and other sciences. One of the most important applications of Shannon's entropy is the detection of the probability model using the principle of maximum entropy, which we describe in this section.

The maximum entropy principle is utilized to analyses the ordinary queueing system by several researchers such as [2] used particle swarm optimization technique to obtain the optimal costs of a reiterate G-queueing framework with working malfunction and working leisure including batch arrival. [3] have conducted an entropy analysis of a flexible Markovian waiting line with server malfunction. Further, Chauhan (2018) has performed a comparative analysis using MEP for the $M^X/(G_1, G_2)/1$ framework with two levels of services. [5] studied an unreliable queueing system with Bernoulli feedback and discouraging behavior of the units arriving at the service system. The relationship between maximum entropy method and queueing theory has been very thoroughly and clearly explained by [6]. Nisha The aspects of general service bulk arrival retrial G-queue including working vacation, state-dependent arrival, priority users, and working breakdown are all explored in this article. Initially, they have estimated performance metrics including orbit size and long-run probabilities in this research work. The maximum entropy approach is then used to give a comparative perusal between the system’s exact and estimated waiting times. Apart from that a bi-objective optimization model is developed to diminish both consumers waiting times and estimated costs simultaneously [7]. Jain and Kaur presented a study of unreliable server retrial bulk queue with multiphase optional service is analyzed by incorporating the features of balking, Bernoulli vacation and Bernoulli feedback. they perform a comparative study of the exact waiting time obtained by the supplementary variable technique and the approximate waiting time derived by using maximum entropy principle by taking the numerical illustration. To verify the outcomes of the model, numerical illustrations and sensitivity analysis have been accomplished [8].

In this paper, we study the N policy M/G/1 queueing system with a removable server using the maximum entropy method; For this purpose, we first provide a summary of queuing theory and the concepts that should be used. Next, we define the entropy function and the maximum entropy principle and describe the method of calculating the maximum entropy distributions. Then we use maximum entropy method to obtain the expected waiting time in the queue; Finally, to see how reliable the principle of maximum entropy is for calculating the interesting measures of the studied systems, we compare the expected waiting time in the queue obtained from the classical method with the corresponding measure obtained from the maximum entropy method for different states of the investigated queuing systems.

In the N policy M/G/1 queueing system with removable server arrivals of customers follow a Poisson process with parameter λ and service times are independent and identically distributed (i.i.d.) random variables obeying a general distribution. The term 'removable server' is just an abbreviation for the system of turning on and turning off the server, depending on the number of customers in the system. The server can be turned on at arrival epochs or off at departure epochs, and during shutdown, it will start providing the service only if the number of applicants in the system is N The total number of arriving customers and the system capacity are infinite. If one customer is in service, then arriving customers have to wait in the queue until the server is available. We assume that customers arrive at the server form a single waiting line and are served in the order of their arrivals; (FCFS discipline). Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers.

1.1 Notations and probabilities

In this research the following notations and probabilities are used.

N: threshold
λ: mean arrival rate.
$\frac{1}{\mu}$: mean service time.
ρ: traffic intensity, where $\rho = \frac{\lambda}{\mu}$. In the steady state $\rho < 1$.
U: service time
S(u): service time distribution function.
E(S) : first moment of the service time distribution
E(S^2): second moment of the service time distribution
E(W): exact expected waiting time in the queue
E(W^*): approximate expected waiting time in the queue
$P_0(0)$: steady-state probability of no customers in the system when the server is turned off
$P_0(n)$: steady-state probability of n customers in the system when the server is turned off
$P_1(n)$: steady-state probability of n customers in the system when the server is turned on and working.
L_{on}:expected number of customers in the system when the server is turned on and working.
L_{off}: expected number of customers in the system when the server is turned off.
L_N: expected number of customers in the system.

2. Problem formulation

Each queuing system has characteristics that can be used to analyze and optimize that system. These characteristics are called interesting measures. Calculating the interesting measures of the system in many cases is very difficult and sometimes impossible, so we have to estimate them. One of these methods is the maximum entropy method. In this research, the N policy M/G/1 queueing system with a removable server was analyzed by using the maximum entropy method.

2.1. Maximum entropy principle

The maximization of H(f) is mostly done on the probability densities off, for entropy maximization we consider the following conditions:

1. $f(x) \geq 0$ (Equality is outside the S)

2. $\int_s^{\cdot} f(x)dx = 1$
3. $\int_s^{\cdot} f(x)r_i(x)dx = \alpha_i \ , 1 \leq i \leq m$

Therefore, f is the density on the space S with the constraints $\alpha_1, \alpha_2, \ldots, \alpha_m$.

We use the method of differential and integral calculus to maximize the entropy function, which is as follows.

We form the following function:

$$J(f(x)) = -\int f(x) \ln f(x)\, dx + \lambda_0 \left(\int f(x)dx - 1\right) + \sum_{i=1}^{m} \lambda_i \left(\int f(x)\, r_i(x)dx - \alpha_i\right)$$

After the derivative we have:

$$\frac{dJ}{df(x)} = -\ln f(x) - 1 + \lambda_0 \sum_{i=1}^{m} \lambda_i\, r_i(x)$$

By setting the above derivative equal to zero, we obtain the following maximized function:

$$f(x) = e^{\lambda_0 - 1 + \sum_{i=1}^{m} \lambda_i r_i(x)} \quad , x \in S$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are chosen so that it applies in the above conditions.

2.2. The maximum entropy formulation for the system

We consider a system Q which has a finite or countable infinite set B of all possible discrete states $B_0, B_1, B_2, \ldots, B_n, \ldots$ Suppose $P(B_n)$ is probability that the system Q is in state B_n. Following [1], we obtain the entropy function as follows:

$$H = -\sum_{B_n \in Q} P(B_n) \ln\{P(B_n)\} \tag{1}$$

which is maximized subject to the following two constraints:

$$\sum_{B_n \in Q} P(B_n) = 1 \tag{2}$$

And

$$\sum_{B_n \in Q} f_k P(B_n) = F_k \quad , \quad k = 1, 2, \ldots, m \tag{3}$$

where Q is the set of queuing system states and F_k is the expected values defined on the set of several suitable functions $\{f_1 P(B_n)\}, \{f_2 P(B_n)\}, \ldots, \{f_m P(B_n)\}$.

The maximum solution of equation (1) under the mentioned constraints is obtained using the Lagrange's method as follows:

$$L = -\sum_{B_n \in Q} P(B_n) \ln\{P(B_n)\} - \beta_0\left(\sum_{B_n \in Q} P(B_n) - 1\right) - \sum_{k=1}^{m} \beta_k \left(\sum_{B_n \in Q} f_k P(B_n) - F_k\right) \tag{4}$$

We derive from this function with respect to $P(B_n)$:

$$\frac{\partial L}{\partial P(B_n)} = -1 - \ln P(B_n) - \beta_0 - \sum_{k=1}^{m} \beta_k f_k(B_n) \tag{5}$$

By putting (5) equal to zero, we have:

$$P(B_n) = \exp[-\beta_0 - 1 - \sum_{k=1}^{m} \beta_k f_k(B_n)] \tag{6}$$

where β_0 is the Lagrange coefficient obtained by normalizing the first constraint, β_k (k=1,2, 3,…,m) is the Lagrange coefficient determined from the second set of constraints.

Now we want to present the maximum entropy solution for the N policy M/G/1 queueing system with a removable server.

2.3. The three basic known constraints

For the N policy M/G/1 queueing system, three well-known constraints are stated as follows:

1. Probability of turning the server off [9]:

$$P_0(0) = P_0(1) = \cdots = P_0(N-1) \tag{7}$$

2. probability that the server is busy [10]:

$$\sum_{n=1}^{\infty} P_1(n) = \rho \tag{8}$$

3. expected number of customers in the system [11]

$$L_N = \frac{N-1}{2} + \lambda\, E(S) + \frac{\lambda^2\, E(S^2)}{2[1-\lambda E(S)]} \tag{9}$$

2.4. The maximum entropy model

In order to develop the steady-state probabilities $P_0(n)$ and $P_1(n)$ by using maximum entropy method, according to (1) and (7)– (9) we formulate the maximum entropy model as follows:

$$H = -\sum_{n=0}^{N-1} P_0(n) \ln P_0(n) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n) \tag{10}$$

or equivalently

$$H = -NP_0(0) \ln P_0(0) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n) \tag{11}$$

Now maximizing (11) subject to the following three constraints:

i. normalizing condition:

$$\sum_{n=0}^{N-1} P_0(n) + \sum_{n=1}^{\infty} P_1(n) = NP_0(0) + \sum_{n=1}^{\infty} P_1(n) = 1 \tag{12}$$

ii. the probability that the server is busy:

$$\sum_{n=1}^{\infty} P_1(n) = \rho \tag{13}$$

iii. the expected number of customers in the system:

$$\sum_{n=0}^{N-1} nP_0(n) + \sum_{n=1}^{\infty} nP_1(n) = \frac{N(N-1)}{2} P_0(0) + \sum_{n=1}^{\infty} nP_1(n) = L_N \tag{14}$$

By multiplying the normalization condition by ω, the probability of the server being busy by θ and the expected number of customers in the system by ϕ, the Lagrange function h can be written as follows (ω, θ and ϕ are the Lagrangian multipliers corresponding):

$$h = -NP_0(0) \ln P_0(0) - \sum_{n=1}^{\infty} P_1(n) \ln P_1(n) - \omega[NP_0(0) + \sum_{n=1}^{\infty} P_1(n) - 1] - \theta[\sum_{n=1}^{\infty} P_1(n) - \rho] - \phi\left[\frac{N(N-1)}{2} P_0(0) + \sum_{n=1}^{\infty} nP_1(n) - L_N\right] \tag{15}$$

The maximum entropy solutions are obtained by taking the partial derivatives of h with respect to $P_0(0)$ and $P_1(n)$, and setting the results equal to zero,

$$\frac{\partial h}{\partial P_0(0)} = -N \ln P_0(0) - N - \omega N - \phi \frac{N(N-1)}{2} = 0 \tag{16}$$

and

$$\frac{\partial h}{\partial P_1(n)} = -\ln P_1(n) - 1 - \omega - \theta - \phi n = 0 \tag{17}$$

It follows from (16) and (17) that we obtain:

$$P_0(0) = e^{-(1+\omega)} e^{\frac{-(N-1)\phi}{2}} \tag{18}$$

and

$$P_1(n) = e^{-(1+\omega)} e^{\phi n} e^{-\theta} \text{ و } n = 1,2,\dots \tag{19}$$

suppose that:

$$\alpha = e^{-(1+\omega)} \quad \text{و} \quad \beta = e^{-\phi} \quad \text{و} \quad \gamma = e^{-\theta}$$

Then $P_0(0)$ and $P_1(n)$ can be written as follows:

$$P_0(0) = \alpha \beta^{\frac{(N-1)}{2}} \tag{20}$$

and

$$P_1(n) = \alpha \beta^n \gamma \quad , \quad n = 1\ ,2, \dots \tag{21}$$

Substituting (20) and (21) into (12)– (14), respectively, yields

$$N \alpha \beta^{\frac{(N-1)}{2}} = 1 - \rho \tag{22}$$

and

$$\sum_{n=1}^{\infty} \alpha \beta^n \gamma = \frac{\alpha \beta \gamma}{1-\beta} = \rho \tag{23}$$

From (22), we get

$$\alpha = \frac{1-\rho}{N} \beta^{\frac{-(N-1)}{2}} \tag{24}$$

After the algebraic manipulations, we obtain c from (23) given by

$$\gamma = \frac{\rho(1-\beta)}{\alpha \beta} \tag{25}$$

Substituting (24) into (25) finally gives

$$\gamma = \frac{N\rho}{1-\rho} \beta^{\frac{N-3}{2}} (1-\beta) \tag{26}$$

after substitute (24) and (26) into (20) and (21), respectively, yielding

$$P_0(0) = \frac{1-\rho}{N} \tag{27}$$

and

$$P_1(n) = \rho(1-\beta)\beta^{n-1} \qquad n = 1, 2, \dots \tag{28}$$

The expected number of customers in the system when the server is turned off

$$L_{off} = \sum_{n=1}^{N-1} n\, P_0(n) = \frac{N(N-1)}{2} P_0(0) = \frac{(N-1)(1-\rho)}{2} \tag{29}$$

Note that $L_N = L_{on} + L_{off}$. From (29), the expected number of customers in the system when the server is turned on and working, L_{on}, is given by

$$L_{on} = \sum_{n=1}^{\infty} nP_1(n) = L_N - L_{off} = L_N - \frac{(N-1)(1-\rho)}{2} \tag{30}$$

substituting (28) into (30) finally gets

$$\beta = 1 - \frac{\rho}{L_{on}} \tag{31}$$

We substitute (31) into (24) to determine the other unknown Lagrangian multiplier a as

$$\alpha = \frac{1-\rho}{N}\left(1 - \frac{\rho}{L_{on}}\right)^{\frac{-(N-1)}{2}} \tag{32}$$

Substituting (31) into (28), we finally obtain

$$P_1(n) = \frac{\rho^2}{L_{on}}\left(1 - \frac{\rho}{L_{on}}\right)^{n-1}, \qquad n = 1و2و\dots \tag{33}$$

The exact expected waiting time in the queue

Using Little's formula, it follows that:

$$E(W) = \frac{1}{\lambda}(L_N - \rho) \tag{34}$$

The approximate expected waiting time in the queue

We define the idle state and the busy state as follows:

I. Idle state denoted by I: the server is turned off and the number of customers waiting in the system is less than N.
II. Busy state denoted by B: the server is busy and provides service to a customer.

We want to get the mean arrival time and mean service time for both I and B modes. Suppose W represents the time that the applicant C waits in the queue to receive service and also suppose that when the customer C enters the system, there are n customers ahead in the queue waiting to receive the service and the system It is in one of I and B states.

I. In idle state I: The server will be turned on after (N-n-1) customers arrive in the system. Thus, customer C will be served until (N-n-1) customers arrive and n customers in front of him waiting for service. The mean arriving time of (N-n-1) customers and the mean service time of n customers is given by $\frac{N-n-1}{\lambda}$ and $\frac{n}{\mu}$ respectively
II. In busy state B: Since the server is turned on, customer C only waits n customers in front of him to be served. The mean service time of n customers is $\frac{n}{\mu}$.

Therefore, the approximate expected waiting time in the queue is equal

$$E(W^*) = \sum_{n=0}^{N-1} \left[\frac{n}{\mu} + \frac{N-n-1}{\lambda}\right] P_0(0) + \sum_{n=0}^{\infty} \frac{n}{\mu} P_1(n) \tag{35}$$

3. Comparision Study

Now, in order to see how accurate the maximum entropy solutions are and whether the maximum entropy method is a reliable method for calculating the effective sizes of the system or not, we compare the expected waiting time in the queue obtained from the classical method with the maximum entropy method. we present specific numerical comparisons between the exact results and the approximate results; For this, we use MATLAB software to calculate the exact expected waiting time in the queue and the approximate expected waiting time in the queue and the relative error percentages.

$$\text{Dev} = \frac{|\text{exact value} - \text{approximate value}|}{\text{exact value}} \times 100 \tag{36}$$

To compare E(W) and $E(W^*)$, we use the following three queuing systems, which are special cases of the N policy M/G/1 queueing system with a removable server:

1) The N policy M/M/1 queueing system with a removable server.

2) The N policy $M/H_2/1$ queueing system with a removable server.

3) The N policy $M/E_2/1$queueing systems with a removable server.

where E_2 is the symbol of Erlang distribution type 2 and H_2 represents hyperexponential type 2.

3.1. The N policy M/M/1 queueing system with a removable server

From [12], we obtain

$$L_N = \frac{N-1}{2} + \frac{\rho}{1-\rho} \tag{37}$$

It implies from (34) that

$$E(W) = \frac{1}{\lambda}\left[\frac{N-1}{2} + \frac{\rho^2}{1-\rho}\right] \tag{38}$$

Substituting (30) and (37) into (33), we finally get

$$P_1(n) = \rho \left[\frac{1}{\frac{N-1}{2} + \frac{1}{1-\rho}}\right] \left[1 - \frac{1}{\frac{N-1}{2} + \frac{1}{1-\rho}}\right]^{n-1}, \qquad n = 1, 2, \ldots \tag{39}$$

Substituting (27) and (39) into (35) yields

$$E(W^*) = \frac{1}{\lambda}\left[\frac{N-1}{2} + \frac{\rho^2}{1-\rho}\right] \tag{40}$$

It is interesting to note that the approximate result $E(W^*)$ obtained in (40) is identical to the exact result $E(W)$ obtained in (38).

We find that the two results are equivalent and using the maximum entropy method exactly achieves the results of the classical methods.

3.2 The N policy M/H_2/1 queueing system with a removable server

From [13] we get

$$L_N = \frac{N-1}{2} + \rho + \frac{q_1\rho_1^2 + q_2\rho_2^2}{1-\rho} \tag{41}$$

Where $q_1 + q_2 = 1$, $\rho_1 = \frac{\lambda}{\mu_1}$, $\rho_2 = \frac{\lambda}{\mu_2}$ and $\rho = q_1\rho_1 + q_2\rho_2$.

From (34) and (41),

$$E(W) = \frac{1}{\lambda}\left[\frac{N-1}{2} + \frac{q_1\rho_1^2 + q_2\rho_2^2}{1-\rho}\right] \tag{42}$$

Substituting (27) and (31) into (35) again, we have

$$E(W^*) = \sum_{n=0}^{N-1}\left[\frac{n}{\mu} + \frac{N-n-1}{\lambda}\right]\frac{1-\rho}{N} + \sum_{n=1}^{\infty}\frac{n}{\mu}\frac{\rho^2}{L_{on}}\left(1 - \frac{\rho}{L_{on}}\right)^{n-1} \tag{43}$$

Where $\frac{1}{\mu} = \frac{q_1}{\mu_1} + \frac{q_2}{\mu_2}$ and L_{on} is given in (30).

We choose $q_1 = 0/3$, $q_2 = 0/7$, $\mu_1 = 0/7$, $\mu_2 = 1$, and varying the values of λ for two cases (i) N = 5 and (ii) N = 10. We perform a comparative analysis for the expected waiting time in the queue for the N policy $M/H_2/1$ queueing system between the approximate results obtained

Table 1. Comparison between the approximate results and the exact results for the N policy $M/H_2/1$ queueing system with a removable server (N=5)

λ	E(W)	$E(W^*)$	Dev (%)
0.1	20.1310	20.1296	0.0065
0.3	7.1842	7.1803	0.0548
0.5	5.2635	5.2570	0.1247
0.7	6.1627	6.1535	0.1491
0.8	9.1786	9.1681	0.1144
0.9	34.5876	34.5758	0.0342

Table 1 has shown that the relative error percentages are less than 1%.

Table 2. Comparison between the approximate results and the exact results for the N policy $M/H_2/1$ queueing system with a removable server (N=10)

λ	E(W)	$E(W^*)$	Dev (%)
0.1	45.1310	45.1296	0.0029
0.3	15.5175	15.5136	0.0254
0.5	10.2635	10.2570	0.0639
0.7	9.7341	9.7249	0.0944
0.8	12.3036	12.2931	0.0853
0.9	37.3654	37.3536	0.0316

Table 2 has shown that the relative error percentages are less than 1%.

3.3. The N policy M/E_2/1queueing systems with a removable server

From [14], we have

$$L_N = \frac{N-1}{2} + \frac{\rho(\rho - k\rho + 2k)}{2k(1-\rho)} \tag{44}$$

Thus we get

$$L_{on} = L_N - L_{off} = L_N - \frac{(N-1)(1-\rho)}{2} \tag{45}$$

From (34), we have

$$E(W) = \frac{1}{\lambda}\left(\frac{(N-1)}{2} + \frac{(1+k)\rho^2}{2k(1-\rho)}\right) \tag{46}$$

Substituting (27) and (33) into (35), it finally gets

$$E(W^*) = \frac{1+\rho}{\lambda} L_N - \frac{1}{\lambda} L_{on} = \frac{\rho}{\lambda} L_N + \frac{1}{\lambda} L_{off} \tag{47}$$

For N policy $M/E_2/1$queueing systems with a removable server, we consider the value of k equal to 2.

In the comparative analysis between E(W) and $E(W^*)$ of the investigated queuing system, we consider the value of μ equal to 1, therefore the value of ρ will be equal to λ.

Table 3. Comparison between the approximate results and the exact results for the N policy $M/E_2/1$ queueing system with a removable server (N=5)

λ= ρ	E(W)	$E(W^*)$	Dev (%)
0.1	20.1083	20.0833	0.1245
0.3	7.0631	6.9881	1.0733
0.5	4.8750	4.7500	2.6316
0.7	4.7821	4.6071	3.7984
0.8	5.7000	5.5000	3.6364
0.9	9.1972	8.9722	2.5077

Table 3 has shown that the relative error percentages are less than 4%.

Table 4. Comparison between the approximate results and the exact results for the N policy $M/E_2/1$ queueing system with a removable server (N=10)

λ= ρ	E(W)	$E(W^*)$	Dev (%)
0.1	45.1083	45.0833	0.0555
0.3	15.3964	15.3214	0.4895
0.5	9.8750	9.7500	1.2821
0.7	8.3536	8.1786	2.1397
0.8	8.8250	8.6250	2.3188
0.9	11.9750	11.7500	1.9149

Table 4 has shown that the relative error percentages are less than 3%.

4. Conclusions

In this paper, we have developed approximate steady-state solutions for the N policy M/G/1 queueing system by using maximum entropy principle. A comparative analysis was made between exact results and approximate results; For this purpose, use three queuing systems which are special cases of the N policy M/G/1 queueing system with a removable server; results has shown that the relative error percentages are very small; Therefore, it can be claimed that the maximum entropy method is sufficiently robust to estimate the interesting measures (service time distribution functions) of the N policy M/G/1 queueing system with a removable server.

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