Computational Sciences and Engineering 4(1) (2024) 67-78

University of Guilan

Computational Sciences and Engineering

journal homepage: https://cse.guilan.ac.ir/

A numerical method for solving a variable-order fractional integral-differential equation

Ahmad Sharif [a,*]

[a] Department of Mathematics, Faculty of Sciences, Gonbad Kavous University, Gonbad, Iran

ARTICLE INFO

Article history:
Received 15 February 2024
Received in revised form 22 March 2024
Accepted 24 March 2024
Available online 1 April 2024

Keywords:
Fractional integral-differential equation
Finite difference method
Numerical solution
Derivative order
Caputo-Prabhakkar
Stability

ABSTRACT

In this paper, a numerical method based on finite differences is presented for numerically solving a fractional integral-differential equation of variable order with Prabhakar integral and Caputo-Prabhakkar fractional derivative. Using the proposed method, an approximate solution of the desired equation is obtained from solving a system of linear equations. The stability of the method is investigated and it is shown that the proposed method is stable under certain conditions. Three examples are presented to demonstrate the efficiency and accuracy of the proposed method.

1. Introduction

In general, differential equations, including integral equations, integral-differential equations, ordinary and partial differential equations, and others, arise from mathematical modeling of real-world problems. We generally have difficulty in obtaining solutions to these types of equations using analytical methods. Therefore, we seek approximate and numerical methods to solve these equations. Research and studies in the theoretical and numerical fields have witnessed many developments in recent decades [1-4]. Most mathematical formulations of physical phenomena include integral-differential equations, which arise in the properties of viscoelasticity [5], risk management models [6], biology [7], and cosmological physics [8]. In the last few decades, by searching through published articles in the fields of mathematical sciences and engineering, we have come across more or less the topics of fractional calculus, integral-differential equations of fractional order, differential equations with fractional derivatives, and similar concepts of this type of topics in fractional calculus [9-13]. These articles and books exist in both theoretical and applied fields and have devoted a significant share of research to themselves. Some authors

* Corresponding author.
E-mail addresses: ah.sharif.edu@gmail.com (A. Sharif)

https://doi.org/10.22124/cse.2025.29847.1094

introduced fractional derivatives and integrals as a generalization of classical concepts and topics and have tried to prove their claim. All these cases indicate that these topics have a good position in mathematical sciences, mathematical physics, and even engineering. In recent years, fractional calculus has made great progress in theoretical and applied fields, which has helped to overcome the shortcomings of integer calculus. Applications of fractional calculus include the study of fluid flow in porous materials, anomalous propagation theory, sound wave propagation in elastic and viscous materials, mechanics of motion of bodies in similar structures, signal processing, financial theory, and electrical conductivity in biological devices. Fractional derivatives have several different definitions, two of the most important of which are the Riemann-Liouville fractional derivative and the Caputo fractional derivative. There is a close relationship between the Riemann-Liouville fractional derivative and the Caputo fractional derivative in that the Riemann-Liouville fractional derivative can be converted to the Caputo fractional derivative under certain functional assumptions [14-16]. In fractional differential equations with partial derivatives, fractional time derivatives are usually defined using the Caputo derivatives. This is because the definition of the Riemann-Liouville fractional derivative requires initial conditions with limit values of the Riemann-Liouville fractional derivative at the origin of time, which do not have very clear physical meanings, while the initial conditions for the Caputo fractional derivative are the same as those for differential equations of the right order.

Some modeling of physics and engineering problems and phenomena will lead to variable-order fractional differential equations [17-19]. In recent years, the application of ordinary variable-order fractional differential equations and variable-order fractional partial differential equations in many fields has increased significantly, and therefore the analysis and solution of these equations is also one of the concerns of researchers. The subject of variable-order fractional differential calculus is actually a generalization of the derivative calculation from natural orders to arbitrary orders. Variable-order fractional differential calculus, due to its many applications in various fields, has attracted the attention of many researchers in the last decade. Variable-order fractional derivatives are a very good tool for describing the memory and hereditary properties of many materials and processes. This is the fundamental advantage of variable-order fractional derivatives compared to ordinary derivatives. The properties of variable-order fractional derivatives are also used in modeling the mechanical and electrical properties of materials in many fields. Variable-order fractional derivatives appear in many physical problems such as frequency-dependent damping behavior in materials, the motion of large thin plates in Newtonian fluids, and control problems in dynamical systems [20-22]. Given the many applications of variable-order fractional derivatives, the solution of variable-order fractional differential equations is of great importance. It is usually not possible to obtain exact solutions to variable-order fractional differential equations, and unlike integer-order differential equations, whose numerical solution has been a fundamental and important topic in numerical and computational mathematics for a long time, there are not many numerical methods for solving variable-order fractional differential equations in general. Usually, numerical methods used to solve functional equations are divided into two categories. The first group is local methods, in which the domain of the problem definition is first divided into a finite number of subdomains, then in each subdomain the solution to the problem is approximated by appropriate basis functions. The finite difference method and the finite element method are among the most important local methods. The finite difference method has good efficiency and accuracy in solving problems with simple domains. The finite element method is also suitable for solving problems whose domains of definition have complex geometric shapes. The second group is comprehensive methods, which approximate the solution to the problems throughout the domain as a collection of appropriate basis functions. Comprehensive methods are usually used to solve problems that have a simple domain structure and shape and require more accuracy. Of course, numerical methods have been proposed for solving linear multivariable fractional differential equations, but there are still not many methods available for solving nonlinear multivariable fractional differential equations. Among them, the implicit Euler numerical method [23], the spline numerical method [24], the Chepishev colocality method of the sixth kind [25], the implicit radial basis function numerical method [26], the numerical method based on matrix operators including Lagrange polynomials [27], the finite difference method [28], and the Legendre wavelet numerical method [29] can be mentioned.

Nowadays, the high-value fractional operator with a non-singular kernel of the Mettig-Leffler function has attracted increasing attention in real-world problems due to its applications in mathematical sciences and engineering. For example, Gara [30], Kielbas [31], Perbahkar [32], the authors generalized the Riemann-Liouville (or Caputo) integral and derivative to the high-value fractional integral and derivative including the generalized Mettig-Leffler function in their kernels. This type of integral and fractional derivative can adequately describe the relaxation time of the unusual Havriliak-Ngami models in the field of dielectric materials [33-35]. Other applications of this type of integral and derivative include the time evolution of polarization processes [36], the fractional Poisson process [30], the Maxwell model in viscoelasticity [37], and the sedimentation of particles in porous media [38].

In this paper, a finite difference numerical method is used to numerically solve a variable fractional order integral-differential equation such as

$$\begin{aligned} &{}^{C}D^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_1(t),\boldsymbol{\omega},\mathbf{0}^+}u(t) + \Im^{\gamma}_{\rho,\mu_2(t),\omega,0^+}u(t) = f(t), \\ &0 < \mu_1(t) \le 1, 0 < \mu_2(t) \le 1, \end{aligned} \tag{1}$$

Under the initial condition

$$u(0) = u_0 \tag{2}$$

is discussed. In relation (1), ${}^{C}D^{\gamma}_{\rho,\mu_1(t),\omega,\mathbf{0}^+}$ the Caputo-multiplicative fractional operator is of variable fractional order $\mu_1(t)$ and $\Im^{\gamma}_{\rho,\mu_2(t),\omega,\mathbf{0}^+}$ the multiplicative fractional operator is of variable fractional order $\mu_2(t)$. For this purpose, the paper is organized as follows. In Section 2, we present important concepts of variable-order integrals and derivatives. In Section 3, we present a finite difference method for solving variable-order integral-differential equations. Also, in this section, the stability of the numerical method is discussed. In Section 4, the proposed method is applied to two examples. Finally, in Section 5, we show the agreement of the presented theorems with the numerical results by solving two examples using the proposed method.

2. Basic concepts in fractional calculus

This section defines the integral and derivative of a multivariable fraction of the order of the variable, which will be used in the following sections.

Definition 1. Let $u \in \mathbf{L}^1(\mathbf{0},\mathbf{b})$, $\mathbf{0} < \mathbf{t} < \mathbf{b} < +\infty$, and $\boldsymbol{\mu}(t)$ be real numbers. In this case, the integral of a multivariable fraction of the order of the variable is defined as follows [32]:

$$(\Im^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t),\boldsymbol{\omega},0^+}\mathbf{u})(\mathbf{t}) = \int_0^t (\mathbf{t} - \tau)^{\boldsymbol{\mu}-1}\, \mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t)}(\boldsymbol{\omega}(\mathbf{t} - \tau)^{\boldsymbol{\rho}})\mathbf{u}\mathbf{d}\tau, \quad \boldsymbol{\mu}(t) \in (0,1], \boldsymbol{\rho} > 0, \tag{3}$$

Which $\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t)}$ is the generalized Mettig-Leffler function [32] introduced by Prabhakar in 1971:

$$\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t)}(\mathbf{z}) = \frac{1}{\Gamma(\boldsymbol{\gamma})}\sum_{\mathbf{n}=0}^{\infty}\frac{\Gamma(\boldsymbol{\gamma}+\mathbf{n})}{\mathbf{n}!\,\Gamma(\boldsymbol{\rho}\mathbf{n}+\boldsymbol{\mu}(t))}\mathbf{z}^{\mathbf{n}}.$$

Definition 2. We assume that the conditions of Definition 1 hold, in which case the derivative of the multiplicative fraction and the Caputo-multiplicative fraction with variable order are defined as follows based on relations (4) and (5), respectively:

$$(\mathbb{D}^{\gamma}_{\rho,\mu(t),\omega,0^+}\mathbf{f})(\mathbf{t}) = \frac{\mathbf{d}}{\mathbf{dt}}\Im^{-\gamma}_{\rho,1-\mu(t),\omega,0^+}u(\mathbf{t}), \mathbf{t} > \mathbf{0}, \tag{4}$$

$$({}^{C}D^{\gamma}_{\rho,\mu(t),\omega,0^+}u)(\mathbf{t}) = \Im^{-\gamma}_{\rho,1-\mu(t),\omega,0^+}\frac{\mathbf{d}}{\mathbf{dt}}u(\mathbf{t}), \mathbf{t} > \mathbf{0}, \tag{5}$$

That $\boldsymbol{\mu}(t) > 0, \boldsymbol{\rho} > \mathbf{0}$.

Lemma 1: Let us assume $\boldsymbol{\rho}, \boldsymbol{\mu}(t)$ that the numbers are real, so that $\boldsymbol{\mu}(t) > 0, \boldsymbol{\rho} > \mathbf{0}$ in this case we have [39]:

$$\int_0^t \tau^{\boldsymbol{\mu}(t)-1}\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t)}(\boldsymbol{\omega}\,\tau^{\boldsymbol{\rho}})\,\mathbf{du} = \mathbf{t}^{\boldsymbol{\mu}(t)}\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}(t)+1}(\boldsymbol{\omega}\,\mathbf{t}^{\boldsymbol{\rho}}). \tag{6}$$

3. Finite difference numerical method for solving equation (1)

In this section, we present a numerical method based on finite differences to solve ***Eq. (1)***. Here, ***Eq. (1)*** is considered on the domain with uniform grid like $0 < t_0 < t_1 < \ldots < t_N = 1$ and $h = t_{j+1} - t_j$. For simplicity, the functions $f(t_j)$ ، $u(t_j)$ and $\mu(t_j)$, are represented as f_j ، u_j and μ_j, respectively. When $0 < \mu_1(t) \leq 1$, then the Caputo-Prabhakkar fractional derivative of $\mu_1(t)$ the variable order is approximated by the finite difference numerical method as follows:

$$\begin{aligned}({}^{C}D^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_1(t),\boldsymbol{\omega},\mathbf{0}^+}u)(\mathbf{t}_{j+1}) &= \Im^{-\boldsymbol{\gamma}}_{\boldsymbol{\rho},\mathbf{1}-\boldsymbol{\mu}_1(t),\boldsymbol{\omega},\mathbf{0}^+}\frac{\mathbf{d}}{\mathbf{dt}}u(\mathbf{t}) \\ &= \int_{\mathbf{0}}^{\mathbf{t}_{j+1}}(\mathbf{t}_{j+1}-\tau)^{-\boldsymbol{\mu}_1(\tau)}\,\mathbf{E}^{-\boldsymbol{\gamma}}_{\boldsymbol{\rho},1-\boldsymbol{\mu}_1(\tau)}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\rho}})\mathbf{u}'\mathbf{d}\tau \\ &\simeq \sum_{k=0}^{j}\frac{u_{k+1}-u_k}{h}\int_{t_k}^{\mathbf{t}_{k+1}}(\mathbf{t}_{j+1}-\tau)^{-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{-\boldsymbol{\gamma}}_{\boldsymbol{\rho},1-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\rho}})\mathbf{d}\tau,\end{aligned} \tag{7}$$

Using ***Eq. (6)***, we have:

$$\begin{aligned}&({}^{C}D^{\gamma}_{\rho,\mu_1(t),\omega,\mathbf{0}^+}u)(\mathbf{t}_{j+1}) \simeq \\ &\sum_{k=0}^{j}\frac{u_{k+1}-u_k}{h}\int_{t_k}^{\mathbf{t}_{k+1}}(\mathbf{t}_{j+1}-\tau)^{-\mu_1(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{-\gamma}_{\rho,1-\mu_1(\frac{t_{k+1}+t_k}{2})}(\omega(\mathbf{t}_{j+1}-\tau)^{\rho})\mathbf{d}\tau \\ &\quad= \sum_{k=0}^{j}\frac{u_{k+1}-u_k}{h}\times[(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{1-\mu_1(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{-\gamma}_{\rho,2-\mu_1(\frac{t_{k+1}+t_k}{2})}(\omega(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\rho}) \\ &\quad-(\mathbf{t}_{j+1}-\mathbf{t}_k)^{1-\mu_1(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{-\gamma}_{\rho,2-\mu_1(\frac{t_{k+1}+t_k}{2})}(\omega(\mathbf{t}_{j+1}-\mathbf{t}_k)^{\rho})].\end{aligned} \tag{8}$$

Similarly, when, $0 < \mu_2(t) \leq 1$ in this case, the integral of a multivalued fraction of variable order $\mu_2(t)$ is obtained using the numerical method of finite differences as

$$\begin{aligned}\Im^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_2(t),\boldsymbol{\omega},\mathbf{0}^+}u(\mathbf{t}) &= \int_{\mathbf{0}}^{\mathbf{t}_{j+1}}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\mu}_2(\tau)-1}\,\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_2(\tau)}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\rho}})\mathbf{ud}\tau \\ &\simeq \sum_{k=0}^{j}\frac{u_{k+1}+u_k}{2}\int_{\mathbf{0}}^{\mathbf{t}_{j+1}}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\mu}_2(\tau)-1}\,\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_2(\tau)}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\rho}})\mathbf{d}\tau \\ &= \sum_{k=0}^{j}\frac{u_{k+1}+u_k}{2}\int_{\mathbf{0}}^{\mathbf{t}_{j+1}}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})-1}\,\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_2(\tau)}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\tau)^{\boldsymbol{\rho}})\mathbf{d}\tau,\end{aligned}$$

is approximated. Using ***Eq. (6)*** we have:

$$\begin{aligned}\Im^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},\boldsymbol{\mu}_2(t),\boldsymbol{\omega},\mathbf{0}^+}u(\mathbf{t}) \simeq \sum_{k=0}^{j}\frac{u_{k+1}+u_k}{2}&\times[(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},1+\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\boldsymbol{\rho}}) \\ &-(\mathbf{t}_{j+1}-\mathbf{t}_k)^{\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}\,\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},1+\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_k)^{\boldsymbol{\rho}})].\end{aligned} \tag{9}$$

Therefore, by substituting ***Eqs. (8)*** and***. (9)*** into ***Eq. (1)***, the discretized ***Eq. (1)*** is obtained as follows:

$$\sum_{k=0}^{j}\Delta_k^j(u_{k+1}-u_k)+\sum_{k=0}^{j}\mathrm{H}_k^j(u_{k+1}+u_k)=f_{j+1}, \\ j=0,1,\ldots,N-1, \tag{10}$$

That

$$\begin{aligned}\Delta_k^j &= (\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{1-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}\mathbf{E}^{-\boldsymbol{\gamma}}_{\boldsymbol{\rho},2-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\boldsymbol{\rho}}) \\ &- (\mathbf{t}_{j+1}-\mathbf{t}_{k})^{1-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}\mathbf{E}^{-\boldsymbol{\gamma}}_{\boldsymbol{\rho},2-\boldsymbol{\mu}_1(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_{k})^{\boldsymbol{\rho}}),\end{aligned} \tag{11}$$

And

$$\begin{aligned}\mathrm{H}_k^j &= (\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},1+\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_{k+1})^{\boldsymbol{\rho}}) \\ &- (\mathbf{t}_{j+1}-\mathbf{t}_{k})^{\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}\mathbf{E}^{\boldsymbol{\gamma}}_{\boldsymbol{\rho},1+\boldsymbol{\mu}_2(\frac{t_{k+1}+t_k}{2})}(\boldsymbol{\omega}(\mathbf{t}_{j+1}-\mathbf{t}_{k})^{\boldsymbol{\rho}}),\end{aligned} \tag{12}$$

For $k=0,1,\ldots,j$. Therefore, ***Eq. (10)*** can be considered as the following system of linear equations:

$$\mathrm{M}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mathrm{H}_0^0-\Delta_0^0 & \mathrm{H}_0^0+\Delta_0^0 & 0 & 0 & 0 & 0 & \cdots \\ \mathrm{H}_0^1-\Delta_0^1 & \mathrm{H}_0^1+\Delta_0^1+\mathrm{H}_1^1-\Delta_1^1 & \Delta_1^1+\mathrm{H}_1^1 & & 0 & 0 & \cdots \\ \mathrm{H}_0^2-\Delta_0^2 & \mathrm{H}_0^2+\Delta_0^2+\mathrm{H}_1^2-\Delta_1^2 & \Delta_1^2+\mathrm{H}_1^2+\mathrm{H}_2^2-\Delta_2^2 & & \Delta_2^2+\mathrm{H}_2^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix} \tag{13}$$

And

$$\mathrm{N}=[u_0,u_1,\ldots,u_N]^T, \\ \mathrm{P}=[h_0,h_1,\ldots,h_N]^T.$$

Now by substituting ***Eqs. (1)*** and ***(9)*** into ***Eq. (1)***, a system of linear equations is obtained. Since it is a lower triangular matrix with non-zero diagonal entries, it is invertible. Therefore, the unknown coefficients are calculated, which means that the numerical solutions of the introduced equation are obtained.

Theorem 1. Let us assume that the coefficients Δ_k^j and H_k^j for $k=0,1,\ldots,j$ and $j=0,1,\ldots,N-1$ introduced in relation (10) hold in relation

$$\mathrm{H}_{j-1}^j+\Delta_{j-1}^j\le 2\Delta_j^j. \tag{14}$$

In this case, the finite difference method (10) is stable.

Proof: It is clear that the coefficients Δ_k^j and H_k^j are both positive. Therefore, we have:

$$2\Delta_j^j - \mathrm{H}_{j-1}^j - \Delta_{j-1}^j \geq 0. \quad \textbf{(15)}$$

Now we can write ***Eq. (10)*** as a recursive relation

$$\begin{aligned} u_{j+1} = \frac{1}{\Delta_j^j + \mathrm{H}_j^j}[f_{j+1} + \Delta_j^j u_j - \mathrm{H}_j^j u_j - \sum_{k=0}^{j-1} \Delta_k^j (u_{k+1} - u_k) \\ - \sum_{k=0}^{j-1} \mathrm{H}_k^j (u_{k+1} + u_k)], \end{aligned} \quad \textbf{(16)}$$

Rewrite. Suppose the error is defined as $\varepsilon_{j+1} = u_{j+1} - \bar{u}_{j+1}$ representing the $\bar{u}_{j+1}$ exact values of the function u in t_{j+1}. Using ***Eq. (16)***, the error in

$$\left|\varepsilon_{j+1}\right| \leq \frac{\Delta_j^j - \mathrm{H}_j^j - \Delta_{j-1}^j - \mathrm{H}_{j-1}^j}{\Delta_j^j + \mathrm{H}_j^j}\left|\varepsilon_j\right|, \quad \textbf{(17)}$$

It is true. By using ***Eq. (15)***, we have

$$\frac{\Delta_j^j - \mathrm{H}_j^j - \Delta_{j-1}^j - \mathrm{H}_{j-1}^j}{\Delta_j^j + \mathrm{H}_j^j} \leq 1. \quad \textbf{(18)}$$

Now, considering ***Eq. (14)***, we have

$$\frac{\Delta_j^j - \mathrm{H}_j^j - \Delta_{j-1}^j - \mathrm{H}_{j-1}^j}{\Delta_j^j + \mathrm{H}_j^j} \geq -1. \quad \textbf{(19)}$$

Therefore, from ***Eqs. (18)*** and ***(19)***, we have

$$\left|\frac{\Delta_j^j - \mathrm{H}_j^j - \Delta_{j-1}^j - \mathrm{H}_{j-1}^j}{\Delta_j^j + \mathrm{H}_j^j}\right| \leq 1. \quad \textbf{(20)}$$

Relation (20) shows that the error is bounded if the method is stable.

4. Numerical Examples

To demonstrate the accuracy and efficiency of the above numerical method, we discuss two numerical examples as follows. These numerical examples are solved for variable-order fractional integral-differential equations with different step lengths, and their numerical solutions are plotted for different step lengths.

Example 1. Consider the following variable-order fractional integral-differential equation:

$$
\begin{aligned}
&{}^{C}D^{\gamma}_{\boldsymbol{\rho},1-0.01t,\boldsymbol{\omega},\mathbf{0}^{+}}u(t) = \Im^{\gamma}_{\rho,\sin^2(t),\omega,0^{+}}u(t) + f(t),\\
&u(0) = 0,\\
&f(t) = \frac{t^{1+\cos^2(t)} - \sqrt{t}s_{\frac{3}{2}+\cos^2(t),\frac{1}{2}}(t)}{\cos^2(t)(1+\cos^2(t))} - \frac{\sqrt{t}s_{\frac{3}{2}-\mu_1(t),\frac{1}{2}}(t) - t^{1-\mu_1(t)}}{\Gamma(2-\mu_1(t))},
\end{aligned}
$$

where $\mu_1(t) = 1 - 0.01t$ and $\mu_2(t) = 1 - \sin^2(t)$. Also $s_{\mu,v}$ the Lommel function defined in [41]. This example was solved using the proposed numerical method for different step lengths and its approximate solution for different values h and the exact solution are shown in ***Figure 1***. We observe that when the step length is reduced, the curve of the approximate solutions approaches the exact solution. ***Table 1*** shows the maximum absolute error between the method of [41] and the proposed method for different values of . It is clearly seen from ***Table 1*** and ***Figure 1*** that the proposed method is an acceptable method to solve this problem.

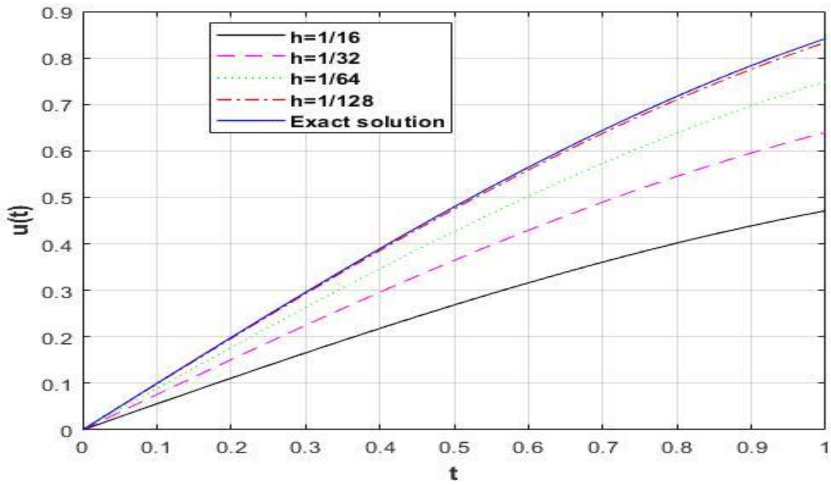

Figure 1. Results from solving Example 1 for different values h.

Table 1. Comparison between the maximum error of the proposed method and the method of [41] for example 1

Method [41]			Suggested method	
N	$\theta = \nu = 0$	$\theta = 0, \nu = -\frac{1}{2}$	h	Maximum absolute error
4	4.7307×10^{-4}	2.9972×10^{-4}	$\frac{1}{32}$	1.3500×10^{-4}
6	1.2874×10^{-6}	6.6335×10^{-7}	$\frac{1}{64}$	3.9600×10^{-5}
8	1.6634×10^{-9}	7.5015×10^{-10}	$\frac{1}{128}$	1.3700×10^{-5}

Example 2. Consider the following fractional-order integral-differential equation of the given variable:

$$ {}^{C}D^{\gamma}_{\rho,\mu_1(t),\omega,0^+}u(t) + \Im^{\gamma}_{\rho,\mu_2(t),\omega,0^+}u(t) = f(t), $$
$$ u(0) = 2, $$

where $\mu_1(t) = t^2$ is a function given so that the exact solution of this $f(t)$ and $\mu_2(t) = \sin(t)$ ، problem becomes $u(t) = \cos(4t) + e^{t^2}$. Using the introduced method, this example is solved for different values of and its approximate solution for different values and the exact solution are shown in ***Figure 2***. We observe that when the step length is reduced, the curve of approximate solutions approaches the exact solution. . ***Table 2*** shows the maximum absolute error between the method [41] and the proposed method for different values of h. It is clearly seen from ***Table 2*** and ***Figure 2*** that the proposed method is an acceptable method for solving this problem.

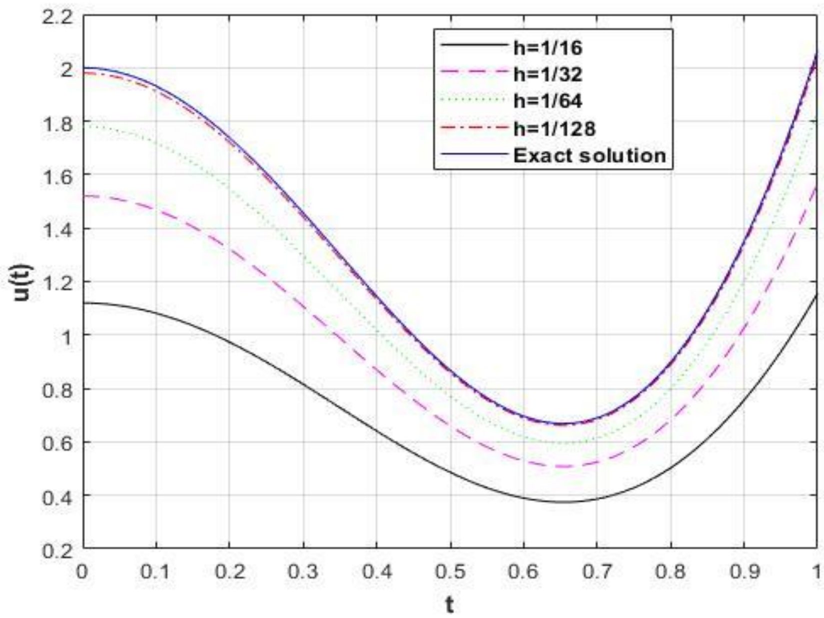

Figure 2. Results from solving Example 2 for different values of h.

Table 2. Comparison between the maximum error of the proposed method and the method of [41] for Example 2

Method [41]			Suggested method	
N	$\theta = \nu = 0$	$\theta = 0, \nu = -\frac{1}{2}$	h	Maximum absolute error
4	2.31×10^{-2}	2.31×10^{-2}	$\frac{1}{32}$	1.3300×10^{-3}
8	9.66×10^{-6}	9.66×10^{-6}	$\frac{1}{64}$	8.2600×10^{-6}
12	4.55×10^{-10}	4.55×10^{-10}	$\frac{1}{128}$	3.3300×10^{-10}
16	1.91×10^{-14}	1.91×10^{-10}	$\frac{1}{256}$	1.500×10^{-10}
20	7.22×10^{-15}	7.22×10^{-15}	$\frac{1}{512}$	6.4400×10^{-15}

Example 3. Consider a variable fractional order integro-differential equation with $\mu_1(t) = t^2 - t + 0.8$,. Using $u(0) = 1$ initial conditions $f(t) = \cos(t^2)$, and $\mu_2(t) = e^{(\sin(5\pi t))}$ the introduced method, this example is solved for different values of h and its approximate solution for different values h and the exact solution are shown in ***Figure 3***. We observe that when the step length is reduced, the curve of approximate solutions approaches the exact solution. . ***Table 3*** shows the absolute error of the proposed method for different values of . It is clearly seen from ***Table 3*** and ***Figure 3*** that the proposed method is an acceptable method for solving this problem.

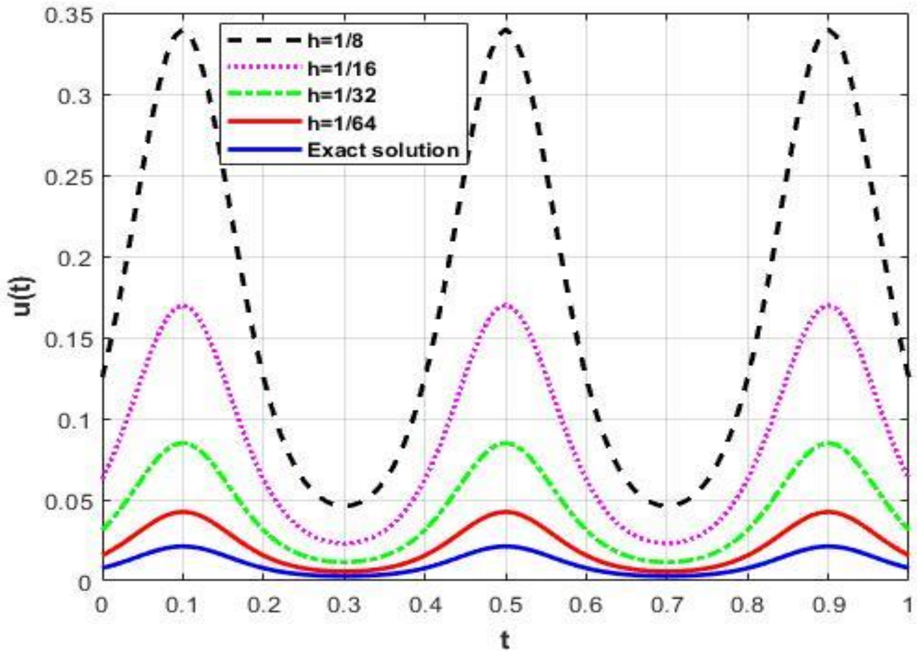

Figure 3. Results from solving Example 3 for different values h .

Table 3. Maximum absolute error of the proposed method for example 3

Step length	Maximum absolute error	Order of approximation
$h = \frac{1}{8}$	0.16495450	-
$h = \frac{1}{16}$	0.10839211	0.6058
$h = \frac{1}{32}$	0.04989323	1.1193
$h = \frac{1}{64}$	0.02202167	1.1799
$h = \frac{1}{128}$	0.00905359	1.2824

5. Conclusion

In this paper, we have presented a numerical method based on the finite difference method for solving variable-order fractional integral-differential equations. The variable-order fractional integral and derivative considered in this paper are multi-function fractional integral and derivative, which are generalizations of Riemann-Liouville and Caputo fractional integral and derivative. We have investigated the stability of the presented method and shown that under certain conditions the method is stable. Three examples were presented to demonstrate the performance of the proposed method, and the numerical results showed that by reducing the step length, the numerical solutions converge to the exact solution with faster growth. The idea of this method can also be considered for other equations in fractional calculus.

References

[1] Volterra V. Theory of functionals and of integral and integro-differential equations. 1959.
[2] Zhao X-Q. Dynamical systems in population biology: Springer; 2003.
[3] Najafi HS, Edalatpanah S, Sheikhani AR. Convergence analysis of modified iterative methods to solve linear systems. Mediterranean journal of mathematics. 2014; 11(3):1019-32.
[4] Najafi HS, Edalatpanah S, Refahisheikhani A. An analytical method as a preconditioning modeling for systems of linear equations. Computational and Applied Mathematics. 2018; 37(2):922-31.
[5] Dehghan M. Solution of a partial integro-differential equation arising from viscoelasticity. International Journal of Computer Mathematics. 2006; 83(1):123-9.
[6] Makroglou A. Integral equations and actuarial risk management: Some models and numerics. Mathematical Modelling and Analysis. 2003; 8(2):143-54.
[7] Hamlin D, Leary R. Methods for using an integro-differential equation as a model of tree height growth. Canadian Journal of Forest Research. 1987; 17(5):353-6.
[8] Mukohyama S. Integro-differential equation for brane-world cosmological perturbations. Physical Review D. 2001; 64(6):064006.
[9] Diethelm K. The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type: Springer Science & Business Media; 2010.
[10] ARDJOUNİ A, DJOUDİ A. Existence and uniqueness of solutions for nonlinear implicit Caputo-Hadamard fractional differential equations with nonlocal conditions. Advances in the Theory of Nonlinear Analysis and its Application. 2019; 3(1):46-52.
[11] Ammi MS, Torres DF. Analysis of fractional integro-differential equations of thermistor type. Handbook of Fractional Calculus with Applications. 2019; 1:327-46.
[12] Mendes EM, Salgado GH, Aguirre LA. Numerical solution of Caputo fractional differential equations with infinity memory effect at initial condition. Communications in Nonlinear Science and Numerical Simulation. 2019; 69:237-47.
[13] Mahboob Dana Z, Najafi HS, Refahi Sheikhani A. An efficient numerical method for solving Benjamin–Bona–Mahony–Burgers equation using difference scheme. Journal of Difference Equations and Applications. 2020:1-12.
[14] Oldham K, Spanier J. The fractional calculus theory and applications of differentiation and integration to arbitrary order: Elsevier; 1974.
[15] Podlubny I. Fractional Differential Equations, Academic Press, San Diego, CA, 1999.
[16] Shariffar F, refahi Sheikhani A, Najafi HS. An efficient chebyshev semi-iterative method for the solution of large systems. University Politehnica of Bucharest Scientific Bulletin- Series A Applied Mathematics and Physics. 2018; 80(4):239-52.
[17] Bahaa G, Abdeljawad T, Jarad F. On Mittag-Leffler Kernel-Dependent Fractional Operators with Variable Order. Fractional Calculus and Fractional Differential Equations: Springer; 2019. p. 41-58.
[18] Abdeljawad T, Mert R, Torres DF. Variable order Mittag–Leffler fractional operators on isolated time scales and application to the calculus of variations. Fractional Derivatives with Mittag-Leffler Kernel: Springer; 2019. p. 35-47.

[19] Sheng H, Sun H, Chen Y, Qiu T. Synthesis of multifractional Gaussian noises based on variable-order fractional operators. Signal Processing. 2011; 91(7):1645-50.
[20] Momani S. A numerical scheme for the solution of multi-order fractional differential equations. Applied Mathematics and Computation. 2006; 182(1):761-70.
[21] Ray SS, Bera R. An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method. Applied Mathematics and Computation. 2005; 167(1):561-71.
[22] Saadatmandi A, Dehghan M. A new operational matrix for solving fractional-order differential equations. Computers & mathematics with applications. 2010; 59(3):1326-36.
[23] Zhuang P, Liu F, Anh V, Turner I. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. SIAM Journal on Numerical Analysis. 2009; 47(3):1760-81.
[24] Yang J, Yao H, Wu B. An efficient numerical method for variable order fractional functional differential equation. Applied Mathematics Letters. 2018; 76:221-6.
[25] Babaei A, Jafari H, Banihashemi S. Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method. Journal of Computational and Applied Mathematics. 2020:112908.
[26] Mohebbi A, Saffarian M. Implicit RBF Meshless Method for the Solution of Two-dimensional Variable Order Fractional Cable Equation. Journal of Applied and Computational Mechanics. 2020 Apr 1; 6(2):235-47.
[27] Solís-Pérez J, Gómez-Aguilar J, Atangana A. Novel numerical method for solving variable-order fractional differential equations with power, exponential and Mittag-Leffler laws. Chaos, Solitons & Fractals. 2018; 114:175-85.
[28] Chen R, Liu F, Anh V. Numerical methods and analysis for a multi-term time–space variable-order fractional advection–diffusion equations and applications. Journal of Computational and Applied Mathematics. 2019; 352:437-52.
[29] Hosseininia M, Heydari MH, Ghaini FM, Avazzadeh Z. A wavelet method to solve nonlinear variable-order time fractional 2D Klein–Gordon equation. Computers & Mathematics with Applications. 2019; 78(12):3713-30.
[30] Garra R, Gorenflo R, Polito F, Tomovski Ž. Hilfer–Prabhakar derivatives and some applications. Applied mathematics and computation. 2014; 242:576-89.
[31] Kilbas AA, Saigo M, Saxena RK. Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms and Special Functions. 2004; 15(1):31-49.
[32] Prabhakar TR. A singular integral equation with a generalized Mittag Leffler function in the kernel. 1971.
[33] Giusti A, Colombaro I, Garra R, Garrappa R, Polito F, Popolizio M, et al. A practical guide to Prabhakar fractional calculus. Fractional Calculus and Applied Analysis. 2020; 23(1):9-54.
[34] Garra R, Garrappa R. The Prabhakar or three parameter Mittag–Leffler function: Theory and application. Communications in Nonlinear Science and Numerical Simulation. 2018; 56:314-29.
[35] Mainardi F, Garrappa R. On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics. Journal of Computational Physics. 2015; 293:70-80.
[36] Garrappa R. Grünwald–Letnikov operators for fractional relaxation in Havriliak–Negami models. Communications in Nonlinear Science and Numerical Simulation. 2016; 38:178-91.
[37] Giusti A, Colombaro I. Prabhakar-like fractional viscoelasticity. Communications in Nonlinear Science and Numerical Simulation. 2018; 56:138-43.
[38] Xu J. Time-fractional particle deposition in porous media. Journal of Physics A: Mathematical and Theoretical. 2017; 50(19):195002.
[39] Suthar DL, Shimelis B, Abeye N, Amsalu H. New composition formulae for the generalized fractional calculus operators with the extended Mittag-Leffler function. Journal| MESA. 2020 May 24; 11(2):309-21.
[40] Ascher UM. Numerical methods for evolutionary differential equations: SIAM; 2008.
[41] Doha EH, Abdelkawy MA, Amin AZ, Lopes AM. On spectral methods for solving variable-order fractional integro-differential equations. Computational and Applied Mathematics. 2018 Jul 1; 37(3):3937-50.