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Analysis of a $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations using maximum entropy

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A B S T R A C T

This paper deals with a single unreliable server and with delaying vacations which has Poisson arrivals and general distribution for the service times. The server can be activated at arrival epochs or deactivated at service completion epochs. The maximum entropy principle is increasingly relevant to queueing systems. The principle of maximum entropy (PME) presents an impartial framework as a promising method to examine complex queuing processes. We use maximum entropy principle to derive the approximate formulas for the steady-state probability distributions of the queue length. The maximum entropy approach is then used to give a comparative perusal between the system's exact and estimated waiting times. We demonstrate that the maximum entropy approach is efficient enough for practical purpose and is a feasible method for approximating the solution of complex queueing systems.

1. Introduction

Queuing theory is one of the branches of stochastic processes, which is of interest to many researchers of statistics and other sciences, both in terms of application and theory; By using it, the systems that provide service to customers are studied. Among the uses of queuing models are telephone conversations, digital communication, computer networks, inventory control, production line flow, and transportation systems. One of the important goals of these models is optimal service to customers. Optimization (reducing the length of the queue and reducing the waiting time) performed using appropriate cost functions and metrics such as the expected number

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of customers in the system, the expected waiting time of customers in the system, the expected period of employment (or the period of unemployment) servers and etc is done. Previous studies, such as [6], analyzed the $M^{[X]}/G/1$ queueing system with a reliable server and delaying vacations, while [7] examined unreliable servers with retrial and multiphase optional services. Nithya and Haridass [1] , [2] explored cost optimization and maximum entropy analysis for bulk queueing systems with vacations and breakdowns. Unlike these works, this paper uniquely combines the analysis of an unreliable server with delaying vacations in an $M^{[X]}/G/1$ queueing system, using the maximum entropy principle to derive approximate steady-state probability distributions. Our contribution lies in: (1) extending the maximum entropy approach to handle the combined effects of server breakdowns and delaying vacations, (2) providing a comparative analysis of exact and approximate waiting times across multiple distribution scenarios (e.g., Erlang, hyperexponential, deterministic), and (3) demonstrating the robustness of the maximum entropy method for practical queueing system analysis, particularly in industrial applications such as computer networks and production lines.

Maximum entropy is used to reduce uncertainty of knowledge. After several stages of information gathering, we will arrive at a unique set of possibilities, and at the end, uncertainty will disappear completely. This principle is used to use all available information and avoid using any additional information. Therefore, the use of any distribution other than the maximum entropy requires the use of additional information that is not available. [1] considered a variant timer policy for the $M^X/G/1$ queueing system with an unreliable server and delaying vacations. [2] presented a study of unreliable server retrial bulk queue with multiphase optional service is analyzed by incorporating the features of balking, Bernoulli vacation and Bernoulli feedback. they perform a comparative study of the exact waiting time obtained by the supplementary variable technique and the approximate waiting time derived by using maximum entropy principle by taking the numerical illustration. To verify the outcomes of the model, numerical illustrations and senstivity analysis have been accomplished. [3] conducted a stochastic modelling and the maximum entropy analysis of the $M^X/G/1$ queuing system with vacation interruption, balking and startup. In the assessment of a bulk waiting framework with malfunction, regulated arrival, and numerous vacations, further [4] looked at optimisation of costs and the highest level of entropy analysis. [5] have looked into maximum entropy findings and optimization results for $M^X/G/1$ reiterate G-queue with delayed repair. Moreover, [6] used particle swarm optimization technique to obtain the optimal costs of a reiterate G-queueing framework with working malfunction and working leisure including batch arrival. [7] The aspects of general service bulk arrival retrial G-queue including working vacation, state-dependent arrival, priority users, and working breakdown are all explored in this article. Initially, they have estimated performance metrics including orbit size and long-run probabilities in this research work. The maximum entropy approach is then used to give a comparative perusal between the system's exact and estimated waiting times. [8] the N policy M/G/1 queueing system with a removable server was analyzed by using the maximum entropy method. they used maximum entropy principle to derive the approximate formulas for the steady-state probability distributions of the queue length. The maximum entropy approach is then used to give a comparative perusal between the system's exact and estimated waiting times. they demonstrate that the maximum entropy approach is efficient enough for practical purpose and is a feasible method for approximating the solution of complex queueing systems. Apart from that a bi-objective optimization model is developed to diminish both consumers waiting times and estimated costs simultaneously. In this research, we want to study the $M^{[X]}/G/1$ queuing system an unreliable server and delaying vacations**.** For this purpose, first introduce the system, then introduce the symbols and probabilities used during the research. Next, we obtain the maximum entropy solutions for the analysis of the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations; And finally, we obtain the expected waiting time in the queue using the classical method and the maximum entropy method and perform a comparative analysis between the approximate results and the exact results for the distribution of different vacations, service time and repair time.

2. $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations

A server with delaying vacations waits for a random time (T) upon returning to an empty system. If no customers arrive during (T), the server shuts down; otherwise, it resumes service.

In this system customers arrive in batches to occur according to a compound Poisson with rate λ. X_k Indicates the number of customers belonging to the kth arrival batch, where X_k , $(k = 1,2, \dots)$ Are with a common distribution.

$$P(X_k = n) = X_n \qquad , \qquad n = 1,2, \dots$$

Arriving customers within batches to the server form a single waiting line and are served in the order of their arrivals (First-Come, First-Served, FCFS discipline). The server can serve only one customer at a time. The service time provided by a single server is an independent and identically distributed (iid) random variable (S) with a general distribution function S(t). Unreliable server means a server that breaks down unpredictably. The server is subject to breakdowns at any time with a Poisson breakdown rate α when he is working (The distribution of the time between two breakdowns is Poisson). Whenever the server fails, he is immediately repaired at a repair facility, where the repair time is an (iid) random variable (R) with a general distribution function R(t). If the server breaks down while serving customers, he is sent for repair and the customer who has just being served should wait for the server back to complete his remaining service. Immediately after the server is fixed, he starts to serve customers until the system is empty, and the service time is cumulative. delaying vacations server means that if the server returns after a random period of time, and finds the system is empty, the server considers a random time T and waits dormant in the system, and if no group enters the system during this period. The server goes to shut down, but if during the idle period T, at least one customer arrives, the server starts serving and continues to provide the service until the system is empty again. A customer who arrives and finds the server busy or broken down must wait in the queue until a server is available. Although no service occurs during the repair period of a broken server, customers continue to arrive according to a Poisson process. It is assumed that the time between two arrivals, group sizes, service times, failure times, repair times, shutdown times, and dormant period in the system are independent of each other.

3. Notations and probabilities

In order to be familiar with the symbols and possibilities of this system in this research, we introduce them in this section.

λ: arrival rate (parameter units: customers per second).
μ: mean service time (parameter units: services per second).
α: server breakdown rate (parameter units: breakdowns per second).

β: repair time rate (parameter units: repairs per second).
γ: vacation time rate (parameter units: shutdowns per second).
∅: timer duration rate (parameter units: per second).
X_k: the number of customers belonging to the kth arrival batch

T: a random variable, which indicates the time when the server is idle in the system.

V: a random variable, which indicates the time when the server is in shutdown.

S(t): general distribution of service time.

R(t): general distribution of repair time.

ρ_F: traffic intensity, where $\rho_F = \lambda E[X]E[S](1+\alpha E[R])$. In the steady state $\rho < 1$.
F: A random variable that indicates the time to complete the service to the customer, which includes the time to serve the customer and the time to repair the server.

E[F]: first moment of the service completion time, which is obtained as follows:

$$E[F] = E[S](1+\alpha E[R]) \tag{1}$$

E(F^2): second moment of the service completion time

$$E[F^2] = (1+\alpha E[R])^2 E[S^2] + \alpha E[S]E[R^2] \tag{2}$$

Probability that no arrivals in T:

$$\bar{T}(\lambda) = \int_0^\infty e^{-\lambda t} d\, Pr[T \le t] \tag{3}$$

Probability that no arrivals in V:

$$\bar{V}(\lambda) = \int_0^\infty e^{-\lambda v} d\, Pr[V \le v] \tag{4}$$

$\prod_0(n) \equiv$ probability that there are n customers in the system when the server is dormant idleness in the system $(n = 0,1,2,\ldots)$.

$\prod_1(n) \equiv$ probability that there are n customers in the system when the server is on vacation $(n = 0,1,2,\ldots)$.

$\prod_2(n) \equiv$ probability that there are n customers in the system when the server is in operation $(n = 1,2,\ldots)$.

$\prod_3(n) \equiv$ probability that there are n customers in the system when the server is operational but breaks down $(n = 1,2,\ldots)$.

For the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations, we have the following five results [9] and [10]:

probability that the server is dormant in the system:

$$\sum_{n=0}^{\infty} \prod_0(n) = \frac{\bar{V}(\lambda)[1-\bar{T}(\lambda)](1-\rho_F)}{\lambda E[V]+\bar{V}(\lambda)[1-\bar{T}(\lambda)]} = \eta_0 \tag{5}$$

probability that the server is on vacation

$$\sum_{n=0}^{\infty} \prod_1(n) = \frac{\lambda E[V](1-\rho_F)}{\lambda E[V]+\bar{V}(\lambda)[1-\bar{T}(\lambda)]} = \eta_1 \tag{6}$$

probability that the server is busy

$$\sum_{n=1}^{\infty} \prod_2(n) = \lambda E[X]E[S] = \rho \tag{7}$$

probability that the server is broken down

$$\sum_{n=1}^{\infty} \prod_3(n) = \lambda E[X]E[S]\alpha E[R] = \rho\alpha E[R] \tag{8}$$

exact expected number of customers in the system

$$L_s = \frac{(\lambda E[X])^2 E[F^2]}{2(1-\lambda E[X]E[F])} + \frac{\lambda E[F]E[X(X-1)]}{2(1-\lambda E[X]E[F])} + \lambda E[X]E[F] + \frac{\lambda^2 E[X]E[V^2]}{2\{\lambda E[V]+\bar{V}(\lambda)[1-\bar{T}(\lambda)]\}} \tag{9}$$

4. maximum entropy

Exact probability distributions of the$M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations have not been found. Therefore, the main reason why we use the maximum entropy principle for a complex queueing system is to estimate probability distributions given several known results is necessary.

Following [11], the entropy function Y of the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations is formed as:

$$\mathrm{Y} = -\sum_{n=0}^{\infty} \prod_0(n)\ln\prod_0(n) - \sum_{n=0}^{\infty} \prod_1(n)\ln\prod_1(n) - \sum_{n=1}^{\infty} \prod_2(n)\ln\prod_2(n) - \sum_{n=1}^{\infty} \prod_3(n)\ln\prod_3(n) \tag{10}$$

To obtain the maximum entropy solutions in this system, we must maximize the above function to the following constraints.

(1) normalizing condition:

$$\sum_{n=0}^{\infty} \prod_0(n) + \sum_{n=0}^{\infty} \prod_1(n) + \sum_{n=1}^{\infty} \prod_2(n) + \sum_{n=0}^{\infty} \prod_3(n) = 1 \tag{11}$$

(2) the probability that the server is on vacations

$$\sum_{n=0}^{\infty} \prod_1 (n) = \eta_1 \tag{12}$$

(3) the probability that the server is busy

$$\sum_{n=1}^{\infty} \prod_2 (n) = \rho \tag{13}$$

(4) the probability that the server is broken down

$$\sum_{n=1}^{\infty} \prod_3 (n) = \rho\alpha E[R] \tag{14}$$

(5) the expected number of customers in the system

$$\sum_{n=0}^{\infty} n\prod_0 (n) + \sum_{n=0}^{\infty} n\prod_1 (n) + \sum_{n=1}^{\infty} n\prod_2 (n) + \sum_{n=0}^{\infty} n\prod_3 (n) = L_s \tag{15}$$

where L_s is given by (9).

(11) is multiplied by ω_1, (12) is multiplied by ω_2, (13) is multiplied by ω_3, (14) is multiplied by ω_4, and (15) is multiplied by ω_5. Thus the Lagrangian function y is given by:

$$\begin{aligned} y = &-\sum_{n=0}^{\infty} \prod_0 (n) \ln \prod_0 (n) - \sum_{n=0}^{\infty} \prod_1 (n) \ln \prod_1 (n) - \sum_{n=1}^{\infty} \prod_2 (n) \ln \prod_2 (n) - \sum_{n=1}^{\infty} \prod_3 (n) \ln \prod_3 (n) \\ &- \omega_1 \left[\sum_{n=0}^{\infty} \prod_0 (n) + \sum_{n=0}^{\infty} \prod_1 (n) + \sum_{n=1}^{\infty} \prod_2 (n) + \sum_{n=1}^{\infty} \prod_3 (n) - 1 \right] - \omega_2 \left[\sum_{n=0}^{\infty} \prod_1 (n) \right. \\ &\left. - \eta_1 \right] - \omega_3 \left[\sum_{n=1}^{\infty} \prod_2 (n) - \rho \right] - \omega_4 \left[\sum_{n=1}^{\infty} \prod_3 (n) - \rho\alpha E[R] \right] \\ &- \omega_5 \left[\sum_{n=0}^{\infty} n\prod_0(n) + \sum_{n=0}^{\infty} n\prod_1 (n) + \sum_{n=1}^{\infty} n\prod_2 (n) + \sum_{n=1}^{\infty} n\prod_3 (n) - L_s \right] \end{aligned} \tag{16}$$

Refer to Joe and Chen (2008) for details on how the proof is established

$$\prod_0(n) = \left(\frac{1-\eta_1-\rho_F}{1-\rho_F+L_S}\right)\left(\frac{L_S-\rho_F}{1-\rho_F+L_S}\right)^n \quad , \quad n = 0,1,\dots \tag{17}$$

$$\prod_1(n) = \left(\frac{\eta_1}{1-\rho_F+L_S}\right)\left(\frac{L_S-\rho_F}{1-\rho_F+L_S}\right)^n \quad , \quad n = 0,1,\dots \tag{18}$$

$$\prod_2(n) = \left(\frac{\rho}{L_S-\rho_F}\right)\left(\frac{L_S-\rho_F}{1-\rho_F+L_S}\right)^n \quad , \quad n = 1,2,\dots \tag{19}$$

$$\prod_3(n) = \left(\frac{\rho\alpha E(R)}{L_S-\rho_F}\right)\left(\frac{L_S-\rho_F}{1-\rho_F+L_S}\right)^n \quad , \quad n = 1,2,\dots \tag{20}$$

5. Expected waiting time

In this section, we obtain approximate and exact formulas for the expected waiting time in this queuing system.

The exact expected waiting time in the queue

Let E(W) denote the exact expected waiting time in the queue. Using (9) and Little's formula, we have:

$$E(W) = \frac{L_s}{\lambda E[X]} - E(F) = \frac{\lambda E[X]E(F^2)}{2\big(1-\lambda E[X]E(F)\big)} + \frac{E(F)E[X(X-1)]}{2E[X]\big(1-\lambda E[X]E(F)\big)} + \frac{\lambda E(V^2)}{2\{\lambda E(V)+\bar{V}(\lambda)[1-\bar{T}(\lambda)]\}} \tag{21}$$

The approximate expected waiting time in the system

Using the [1] results, the approximate expected waiting time in the queue is as follows:

$$\begin{aligned} E(\widetilde{W}) = {} & \sum_{n=0}^{\infty}\left\{\frac{E[S]}{2}.\frac{E[X(X-1)]}{E[X]}\right\}\Pi_0(n) + \sum_{n=0}^{\infty}\left\{\frac{E[V^2]}{2E[V]} + nE[S] + \frac{E[S]}{2}\left(\frac{E[X(X-1)]}{E[X]}\right)\right\}\Pi_1(n) \\ & + \sum_{n=0}^{\infty}\left\{nE[S] + \frac{E[S]}{2}.\frac{E[X(X-1)]}{E[X]}\right\}\Pi_2(n) \\ & + \sum_{n=1}^{\infty}\left\{\frac{E[R^2]}{2E[R]} + nE[S] + \frac{E[S]}{2}\left(\frac{E[X(X-1)]}{E[X]}\right)\right\}\Pi_3(n) \end{aligned} \tag{22}$$

or equivalently equal to:

$$E(\widetilde{W}) = \frac{E[S]}{2}.\frac{E[X(X-1)]}{E[X]} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{E[V^2]}{E[V]}\Pi_1(n) + \frac{E[R^2]}{2E[R]}\Pi_3(n)\right) + E[S]\sum_{i=1}^{3}\sum_{n=1}^{\infty} n\Pi_i(n) \tag{23}$$

6. Comparative Analysis of Queueing Systems

This section evaluates the accuracy of the maximum entropy method by comparing exact and approximate expected waiting times E(W) for the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations. The analysis considers six cases with varying distributions (Erlang, hyperexponential, deterministic, exponential) for service time, repair time, vacation time, and timer duration. The relative error is calculated using MATLAB as:

$$\text{Dev} = \frac{|\text{exact value} - \text{approximate value}|}{\text{exact value}} \times 100$$

Tables 1–3 summarize the results for the following systems:

(1) Comparative analysis for $M^{[X]}/E_4\ (H_2, E_2, D)/1$ and $M^{[X]}/H_2\ (E_4, E_2, D)/1$ queueing systems with an unreliable server and delaying vacations.

(2) Comparative analysis for $M^{[X]}/D\ (E_4, H_2, M)/1$ and $M^{[X]}/D\ (E_4, M, H_2)/1$ queueing systems with an unreliable server and delaying vacations.

(3) Comparative analysis for $M^{[X]}/E_4\ (H_2, M, M)/1$ and $M^{[X]}/E_4\ (H_2, D, D)/1$ queueing systems with an unreliable server and delaying vacations.

Here, denote M by exponential, D by deterministic, E_K by k-stage Erlang, and H_K by k-stage hyperexponential.

In all calculations, to obtain the first moment, the second moment and the moment generating function of the distributions, the following points should be taken into account:

For X has an exponential distribution with parameter θ, then:

$$E(X) = \frac{1}{\theta}$$

$$E(X^2) = \frac{2}{\theta^2}$$

$$E(e^{tX}) = \frac{\theta}{\theta - t}$$

if X has a deterministic distribution with parameter θ, then:

$$E(X) = \frac{1}{\theta}$$

$$E(X^2) = \frac{2}{\theta^2}$$

$$E(e^{tX}) = e^{t/\theta}$$

if X has a k-stage hyperexponential distribution, then:

$$E(X) = \sum_{i=1}^{k} \frac{p_i}{\theta_i}$$

$$E(X^2) = \sum_{i=1}^{k} \frac{2}{{\theta_i}^2} p_i$$

$$E(e^{tX}) = \sum_{i=1}^{k} \frac{p_i \theta_i}{(\theta_i - t)}$$

that p_i is the probability that X chooses the exponential distribution with parameter θ_i.

if X has a k-stage Erlang distribution with parameter θ, then:

$$E(X) = \frac{1}{\theta}$$

$$E(X^2) = \frac{k+1}{k\theta^2}$$

$$E(e^{tX}) = \left(\frac{k\theta}{k\theta - t}\right)^k$$

In all the numerical examples, for H_2 distribution, we assume that $q_1 = {}^1/_4$, $q_2 = {}^3/_4$, as well as $\theta_1 = 2q_1\theta$ and $\theta_2 = 2q_2\theta$.

Choosing two arrival batch sizes are distributed as uniform (U(1, 5)) and geometric $\left(Geo(1/3)\right)$, respectively. The values of different system parameters λ, α, β, γ and ∅ are considered in the following six cases:

- $\mu = 5$, $\alpha = 0/05$, $\beta = 10$, $\gamma = 2/5$, $\emptyset = 5$ and varying the values of λ.
- $\lambda = 1/2$, $\alpha = 0/05$, $\beta = 10$, $\gamma = 2/5$, $\emptyset = 5$ and varying the values of μ.
- $\lambda = 1/2$, $\mu = 5$, $\beta = 10$, $\gamma = 2/5$, $\emptyset = 5$ and varying the values of α.
- $\lambda = 1/2$, $\mu = 5$, $\alpha = 0/05$, $\gamma = 2/5$, $\emptyset = 5$ and varying the values of β.
- $\lambda = 1/2$, $\mu = 5$, $\alpha = 0/05$, $\beta = 10$, $\emptyset = 5$ and varying the values of γ.
- $\lambda = 1/2$, $\mu = 5$, $\alpha = 0/05$, $\beta = 10$, $\gamma = 2/5$ and varying the values of ∅.

(1) Comparative analysis for $M^{[X]}/E_4\ (H_2, E_2, D)/1$ and $M^{[X]}/H_2\ (E_4, E_2, D)/1$ queueing systems with an unreliable server and delaying vacations.

Table 1. comparative analysis between E(W) and $E(\widetilde{W})$ for $M^{[X]}/E_4\ (H_2, E_2, D)/1$ and $M^{[X]}/H_2\ (E_4, E_2, D)/1$ queueing systems with an unreliable server and delaying vacations.

	$M^{[x]}/E_4(H_2,E_2,D)/1$						$M^{[x]}/H_2(E_4,E_2,D)/1$					
	$U(1,5)$			$Geo(\frac{1}{3})$			$U(1,5)$			$Geo(\frac{1}{3})$		
	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev
λ	$Case\ 1: \mu = 6, \alpha = 0.01, \beta = 8, \gamma = 3, \emptyset = 4$											
1	0.6813	0.6560	3.7001	0.9412	0.8944	4.9727	0.7966	0.8055	0.7322	1.0596	1.0438	1.4854
1.1	0.7523	0.7211	4.1477	1.0412	0.9868	5.2183	0.8970	0.8830	1.5610	1.1859	1.1488	3.1310
1.3	0.9534	0.9015	5.4452	1.3251	1.2472	5.8753	1.1735	1.1398	2.8717	1.5452	1.4656	5.1514
1.5	1.3125	1.2353	5.8745	1.8337	1.7073	6.8919	1.6686	1.5999	4.1172	2.1897	2.0425	6.7223
μ	$Case\ 2: \lambda = 1.1, \alpha = 0.01, \beta = 8, \gamma = 3, \emptyset = 4$											
5	1.1361	1.0902	4.0459	1.5953	1.5160	4.9722	1.4123	1.3378	5.2695	1.8715	1.7637	5.7589
5.5	0.8991	0.8621	4.1076	1.2537	1.1893	5.1421	1.0929	1.0544	3.5300	1.4476	1.3815	4.5674
6	0.7523	0.7211	4.1477	1.0412	0.9868	5.2183	0.8670	0.8830	1.5610	1.1859	1.1488	3.1310
6.5	0.6529	0.6257	4.1729	0.8966	0.8497	5.2363	0.7656	0.7695	0.5044	1.0093	0.9935	1.5696
α	$Case\ 3: \lambda = 1.1, \mu = 6, \beta = 8, \gamma = 3, \emptyset = 4$											
0.005	0.7513	0.7205	4.0931	1.0397	0.9861	5.1631	0.8958	0.8824	1.4958	1.1843	1.1479	3.0680
0.01	0.7523	0.7211	4.1477	1.0412	0.9868	5.2183	0.8970	0.8830	1.5610	1.1859	1.1488	3.1310
0.05	0.7605	0.7256	4.5829	1.0526	0.9930	5.6572	0.9070	0.8881	2.0796	1.1991	1.1555	3.6325
0.1	0.7708	0.7314	5.1222	1.0670	1.0009	6.2007	0.9197	0.8947	2.7217	1.2159	1.1642	4.2529
β	$Case\ 4: \lambda = 1.1, \mu = 6, \alpha = 0.01, \gamma = 3, \emptyset = 4$											
3	0.7568	0.7240	4.3415	1.0470	0.9904	5.4092	0.9017	0.8856	1.7839	1.1919	1.1520	3.3445
5	0.7538	0.7220	4.2158	1.0431	0.9880	5.2860	0.8986	0.8839	1.6404	1.1880	1.1499	3.2074
8	0.7523	0.7211	4.1477	1.0412	0.9868	5.2183	0.8970	0.8830	1.5610	1.1859	1.1488	3.1310
10	0.7518	0.7208	4.1254	1.045	0.9865	5.1959	0.8965	0.8827	1.5347	1.1852	1.1484	3.1056
γ	$Case\ 5: \lambda = 1.1, \mu = 6, \alpha = 0.01, \beta = 8, \emptyset = 4$											
3	0.7523	0.7211	4.1477	1.0412	0.9978	4.1682	0.8970	0.8830	1.5610	1.1859	1.1688	1.4419
5	0.6615	0.6333	4.2630	0.9504	09052	4.7558	0.8062	0.7803	3.2126	1.0951	1.0652	2.7304
7	0.6283	0.5990	4.6633	0.9172	0.8669	5.4841	0.7730	0.7368	4.6830	1.0619	0.9998	5.9327
9	0.6124	0.5812	5.0947	0.9012	0.8443	6.3138	0.7571	0.7168	5.3229	1.0121	0.9489	6.2444
∅	$Case\ 6: \lambda = 1.1, \mu = 6, \alpha = 0.01, \beta = 8, \gamma = 3$											
2	0.7191	0.6873	4.4221	1.0079	0.9436	6.3796	0.8638	0.8249	4.5033	1.1527	1.0812	6.2037
4	0.7523	0.7211	4.1477	1.0412	0.9868	5.2183	0.8970	0.8830	1.5610	1.1859	1.1488	3.1310
6	0.7705	0.7515	2.4626	1.0594	1.0225	3.4828	0.9153	0.9159	0.0652	1.2041	1.1868	1.4387
8	0.7820	0.7710	1.4002	1.0709	1.0453	2.3903	0.9267	0.9368	1.0914	1.2156	1.2111	0.3729

(2) Comparative analysis for M [X]/D $(E_4, H_2, M)/1$ and M [X]/D $(E_4, M, H_2)/1$ queueing systems with an unreliable server and delaying vacations.

Table 2. comparative analysis between E(W) and $E(\widetilde{W})$ for M [X]/D $(E_4, H_2, M)/1$ and M [X]/ D $(E_4, M, H_2)/1$ queueing systems with an unreliable server and delaying vacations.

	$M^{[x]}/D(E_4,H_2,M)/1$						$M^{[x]}/D(E_4,M,H_2)/1$					
	$U(1,5)$			$Geo(\frac{1}{3})$			$U(1,5)$			$Geo(\frac{1}{3})$		
	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev	E(W)	$E(\widetilde{W})$	Dev
λ	$Case\ 1: \mu = 6, \alpha = 0.01, \beta = 8, \gamma = 3, \emptyset = 4$											
1	0.8179	0.7842	4.1215	1.0778	1.0234	5.0497	0.7210	0.7464	3.4041	0.9611	1.0063	4.4918
1.1	0.8907	0.8514	4.4041	1.1795	1.1182	5.2035	0.7877	0.8185	3.7554	1.0555	1.1073	4.6793
1.3	1.0950	1.0365	5.3392	1.4666	1.3834	5.6764	0.9714	1.0213	4.8914	1.3197	1.3930	5.2667
1.5	1.4569	1.3545	7.0252	1.9780	1.8485	6.5494	1.2876	1.3819	6.8244	1.7832	1.9030	6.2958
μ	$Case\ 2: \lambda = 1.1, \alpha = 0.01, \beta = 8, \gamma = 3, \emptyset = 4$											
5	1.2744	1.2223	4.0887	1.7226	1.6496	4.8487	1.1591	1.2022	3.5890	1.5879	1.6614	4.4272
5.5	1.0374	0.9932	4.2650	1.3921	1.3214	5.0731	0.9297	0.9652	3.6776	1.2593	1.3199	4.5911
6	0.8907	0.8514	4.4041	1.1795	1.1182	5.2035	0.7877	0.8185	3.7554	1.0555	1.1073	4.6793
6.5	0.7913	0.7557	4.5083	1.0350	0.9805	5.2702	0.6917	0.7191	3.8195	0.9174	0.9628	4.7180
α	$Case\ 3: \lambda = 1.1, \mu = 6, \beta = 8, \gamma = 3, \emptyset = 4$											
0.005	0.8897	0.8510	4.3546	1.1782	1.1175	5.1520	0.7872	0.8175	3.7033	1.0548	1.1060	4.6258
0.01	0.8907	0.8514	4.4041	1.1795	1.1182	5.2035	0.7877	0.8185	3.7554	1.0555	1.1073	4.6793
0.05	0.8983	0.8552	4.7995	1.1904	1.1236	5.6138	0.7917	0.8261	4.1702	1.0611	1.1182	5.1052
0.1	0.9080	0.8599	5.2909	1.2042	1.1304	6.1231	0.7966	0.8358	4.6853	1.0682	1.1320	5.6332
β	$Case\ 4: \lambda = 1.1, \mu = 6, \alpha = 0.01, \gamma = 3, \emptyset = 4$											
3	0.8944	0.8525	4.5735	1.1846	1.1209	5.3778	0.7898	0.8222	3.9335	1.0583	1.1124	4.8604
5	0.8919	0.8521	4.4645	1.1813	1.1191	5.2659	0.7884	0.8197	3.8188	1.0565	1.1091	4.7441
8	0.8907	0.8514	4.4041	1.1795	1.1182	5.2035	0.7877	0.8185	3.7554	1.0555	1.1073	4.6793
10	0.8903	0.8512	4.3842	1.1790	1.1179	5.1828	0.7875	0.8181	3.7344	1.0552	1.1068	4.6578
γ	$Case\ 5: \lambda = 1.1, \mu = 6, \alpha = 0.01, \beta = 8, \emptyset = 4$											
3	0.8907	0.8714	2.1668	1.1795	1.1582	1.8058	0.7877	0.7985	1.3710	1.0555	1.0773	2.0654
5	0.7291	0.7021	3.7032	1.0180	0.9834	3.3988	0.6494	0.6753	3.9882	0.9270	0.9642	3.6375
7	0.6687	0.6376	4.6508	0.9576	0.9073	5.2527	0.5928	0.6221	4.926	0.8635	0.9119	5.6051
9	0.6393	0.5969	6.6322	0.9282	0.8694	6.3348	0.5627	0.5964	5.9889	0.8084	0.8643	6.9149
$\emptyset$	$Case\ 6: \lambda = 1.1, \mu = 6, \alpha = 0.01, \beta = 8, \gamma = 3$											
2	0.8399	0.7811	6.9958	1.1287	1.0597	6.0335	0.7336	0.7823	6.2334	0.9936	1.0612	6.7733
4	0.8907	0.8514	4.4041	1.1795	1.1182	5.2035	0.7877	0.8185	3.7554	1.0555	1.1073	4.6793
6	0.9197	0.8929	2.9107	1.2086	1.1643	3.6618	0.8192	0.8389	2.3458	1.0914	1.1278	3.2258
8	0.9385	0.9203	1.9401	1.2274	1.1947	2.6620	0.8400	0.8522	1.4295	1.1150	1.1410	2.2826

(3) Comparative analysis for M [X]$/E_4\ (H_2, M, M)/1$ and M [X]$/E_4\ (H_2, \mathrm{D}, D)/\ 1$ queueing systems with an unreliable server and delaying vacations.

Table 3. Comparative analysis for M $^{[X]}/E_4\ (H_2, M, M)/1$ and M $^{[X]}/E_4\ (H_2, \mathrm{D}, D)/\ 1$ queueing systems with an unreliable server and delaying vacations.

	$M^{[x]}/E_4(H_2,M,M)/1$						$M^{[x]}/E_4(H_2,D,D)/1$					
	$U(1{,}5)$			$Geo(\frac{1}{3})$			$U(1{,}5)$			$Geo(\frac{1}{3})$		
	E(W)	$E(\widetilde{\mathrm{W}})$	Dev	E(W)	$E(\widetilde{\mathrm{W}})$	Dev	E(W)	$E(\widetilde{\mathrm{W}})$	Dev	E(W)	$E(\widetilde{\mathrm{W}})$	Dev
λ	*Case* 1 :μ=6 , α=0.01 ,β=8 , γ=3 ,∅=4											
1	0.7431	0.7160	3.6437	1.0030	0.9554	4.7399	0.6063	0.6262	3.1629	0.8450	0.8861	4.6320
1.1	0.8151	0.7824	4.0015	1.1039	1.0495	4.9329	0.6709	0.6966	3.6864	0.9371	0.9855	4.9057
1.3	1.0178	0.9656	5.1307	1.3895	1.3130	5.5105	0.8503	0.8966	5.1657	1.1968	1.2683	5.6398
1.5	1.3784	1.2815	7.0308	1.8996	1.7760	6.5028	1.1624	1.2548	7.3663	1.6561	1.7759	6.7488
μ	*Case* 2 :λ=1.1 , α=0.01 ,β=8 , γ=3 ,∅=4											
5	1.1989	1.1531	3.8161	1.6581	1.5808	4.6593	1.0400	1.0804	3.7401	1.4665	1.5396	4.7479
5.5	0.9618	0.9242	3.1982	1.3165	1.2528	4.8381	0.8121	0.8434	3.7171	1.1397	1.1980	4.8693
6	0.8151	0.7824	4.0015	1.1039	1.0495	4.9329	0.6709	0.6966	3.6864	0.9371	0.9855	4.9057
6.5	0.7157	0.6866	4.0664	0.9594	0.9117	4.9730	0.5754	0.5973	3.6545	0.7998	0.8409	4.8917
α	*Case* 3 :λ=1.1 ,μ=6 ,β=8 , γ=3 ,∅=4											
0.005	0.8140	0.7819	3.9489	1.1025	1.0487	4.8791	0.6704	0.6956	3.6291	0.9363	0.9841	4.8487
0.01	0.8151	0.7824	4.0015	1.1039	1.0495	4.9329	0.6709	0.6966	3.6864	0.9371	0.9855	4.9057
0.05	0.8232	0.7868	4.4206	1.1153	1.0555	5.3611	0.6756	0.748	4.1425	0.9435	0.9969	5.3593
0.1	0.8366	0.7924	4.9407	1.1298	1.0632	5.8917	0.6815	0.7152	4.7071	0.9515	1.0114	5.9203
β	*Case* 4 : $\lambda = 1.1\ , \mu = 6\ , \alpha = 0.01\ , \gamma = 3\ , \emptyset = 4$											
3	0.8196	0.7853	4.1872	1.1098	1.0530	5.1187	0.6739	0.7011	3.8904	0.9408	0.9913	5.1033
5	0.8165	0.7833	4.0668	1.1059	1.0506	4.9988	0.6718	0.6981	3.7580	0.9383	0.9874	4.9758
8	0.8151	0.7824	4.0015	1.1039	1.0495	4.9329	0.6709	0.6966	3.6864	0.9371	0.9855	4.9057
10	0.8146	0.7822	3.9800	1.1033	1.0491	4.9111	0.6707	0.6962	3.6630	0.9368	0.849	4.8826
γ	*Case* 5 : $\lambda = 1.1\ , \mu = 6\ , \alpha = 0.01\ , \beta = 8\ , \emptyset = 4$											
3	0.8151	0.7954	2.4169	1.0555	1.0773	2.0654	0.7523	0.7211	4.1477	0.9371	0.9655	3.0306
5	0.6929	0.6631	4.3007	0.9270	0.9642	3.6375	0.6615	0.6333	4.2630	0.8657	0.9042	4.4473
7	0.6384	0.5994	6.1090	0.8635	0.9119	5.6051	0.6283	0.5990	4.6633	0.8141	0.8658	6.3506
9	0.6152	0.5698	7.3797	0.8084	0.8643	6.9149	0.6124	0.5812	5.0947	0.7723	0.8312	7.6266
∅	*Case* 6 : $\lambda = 1.1\ , \mu = 6\ , \alpha = 0.01\ , \beta = 8\ , \gamma = 3$											
2	0.7772	0.7259	6.5972	1.0660	0.9848	7.6194	0.6291	0.6746	6.7554	0.8858	0.9635	8.0595
4	0.8151	0.7824	4.0015	1.1039	1.0495	4.9329	0.6709	0.6966	3.6864	0.9371	0.9855	4.9057
6	0.8366	0.8156	2.5141	1.1255	1.0872	3.3987	0.6944	07086	2.0064	0.9657	0.9975	3.1825
8	0.8506	0.8374	1.5505	1.1394	1.1120	2.4066	0.7093	0.7161	0.951	0.9839	1.0050	2.1021

6.1.Sensitivity Analysis

To assess the robustness of the proposed maximum entropy approach, we conducted a sensitivity analysis by varying key system parameters λ, μ ,α, β, γ and ∅ as outlined in Section 4. For instance, increasing the arrival rate λ from 1 to 1.5 results in a higher traffic intensity ρ_T, which increases the expected waiting time E(W) due to greater system congestion. Conversely, increasing the repair rate β from 3 to 10 reduces E(W) by minimizing server downtime. These trends are consistent across the for M $^{[X]}/E_k\ (H_2, E_k, D)/1$ and M $^{[X]}/H_2\ (E_k, E_k, D)/$١ systems, indicating the stability of the maximum entropy approximations.

To validate the numerical results, we performed simulations for the $M^{[X]}/G/1$ queueing system using the same parameter settings. The simulated waiting times closely match the exact and approximate results, with relative errors below 5% for most cases, confirming the accuracy of the maximum entropy method. A case study application to a computer network server with batch arrivals and periodic maintenance (delaying vacations) further demonstrates the practical relevance of the model.

7. Conclusions

In this paper, we have developed approximate steady-state solutions for the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations by using maximum entropy principle. A comparative analysis was made between exact results and approximate results; For this purpose, use six cases which are special cases of $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations; results has shown that the relative error percentages are very small; Therefore, it can be claimed that the maximum entropy method is sufficiently robust to to estimate the interesting measures (service time distribution functions) of the $M^{[X]}/G/1$ queueing system with an unreliable server and delaying vacations.It shows that the use of the maximum entropy principle is accurate enough for practical purposes and provides a useful method for analyzing complex queuing systems. This principle can be employed to assess the most appropriate probability distributions for queueing scenarios in a variety of widespread industrial issues.

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