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Statistical Inference for the Lindley-Exponential Distribution Using Lower Record Values

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ABSTRACT

This paper presents a comprehensive study on statistical inference for the Lindley-Exponential (LE) distribution based on lower record values. We derive key distributional properties of the LE model, including the density and moments of lower record statistics. Both classical and Bayesian frameworks are developed for parameter estimation. The maximum likelihood method is employed to obtain point estimates and asymptotic confidence intervals. For the Bayesian approach, independent gamma priors are assumed for the parameters, and estimation is conducted under symmetric (squared error) and asymmetric (LINEX) loss functions. Since the posterior distributions are analytically intractable, we use the Tierney–Kadane approximation and a Metropolis–Hastings algorithm within the Gibbs framework. Furthermore, we address the problem of predicting future lower record values using both maximum likelihood and Bayesian predictive distributions. Extensive Monte Carlo simulations are conducted to evaluate the performance of the proposed estimators and predictors. The results indicate that the Bayesian estimators under squared error loss often yield lower expected risks, and the predictive accuracy improves with the number of observed records. The methodologies developed in this study are particularly useful for modeling and predicting extreme or record-breaking events in fields such as reliability engineering, meteorology, and economics.

1. Introduction

The probability density function (PDF) of the Lindley–Exponential (LE) distribution is defined as

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$$f(x;\theta) = \frac{\lambda\theta^2}{1+\theta}(1+\lambda x)e^{-\lambda\theta x}, x > 0, \theta > 0. \tag{1}$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x;\theta) = 1 - \left(1 + \frac{\lambda\theta x}{\theta+1}\right)e^{-\lambda\theta x}. \tag{2}$$

The Lindley–Exponential distribution was first proposed by Cakmakyapan and Ozel [1] as a novel lifetime model capable of describing various reliability behaviors. The distribution with parameters θ and λ is denoted by $LE(\theta, \lambda)$. When $\lambda = 1$, the model reduces to the standard Lindley distribution.

The survival function of the LE distribution is expressed as

$$S(x;\theta) = \left(1 + \frac{\lambda\theta x}{\theta+1}\right)e^{-\lambda\theta x}. \tag{3}$$

and the corresponding hazard rate function is

$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda\theta^2(1+\lambda x)}{\theta+1+\lambda\theta x}. \tag{4}$$

Let $X_n, n = 1,2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables. An observation X_j is said to be a lower record value if $X_j < X_i$ for all $i < j$. The sequence of lower record values and inter-record times can be denoted by $(l, t) = (l_1, t_1), (l_2, t_2), \ldots, (l_r, t_{r-1})$ where l_i represents the i-th lower record value and t_i denotes the inter-record time—the number of observations between l_i and the next record l_{i+1}.

Because of their significance in many applied sciences, record values have attracted considerable attention in statistical literature. Numerous studies and monographs have explored their properties and inferential aspects, including works by Chandler [2], Resnick [3], Shorrock [4], Glick [5], Nevzorov [6], Ahsanullah and Nevzorov [7], Balakrishnan and Ahsanullah [8], Arnold et al. [9], Dey and Dey [10], Dey et al. [11], Kızılaslan and Nadar [12-14], Tarvirdizade and Ahmadpour [15], Ahmadi and Doostparast [16], Doostparast and Balakrishnan [17], Zanjiran and Mirmostafaee [17], Pak and Dey [19], Dey et al. [20], Amiri and MirMostafaee [21-22], and Bastan and MirMostafaee [23].

Previous studies have focused on the estimation and prediction of the parameters of the Lindley–Exponential (LE) distribution using complete data. In the present study, we concentrate on the estimation of model parameters based on record data and also investigate the prediction of future record values arising from the Lindley–Exponential distribution.

One of the main motivations for the authors to focus on the analysis of record data is the emphasis on extreme values (maximum or minimum) and the prediction of rare events. In many statistical applications, such as reliability analysis, risk assessment, and industrial or environmental data, extreme and exceptional observations are often more informative than the rest of the data. Record data, due to their unique statistical properties—such as the independence of inter-record times and their focus on the critical points of the distribution—allow for more accurate parameter estimation and future prediction. Furthermore, the analysis of record values is particularly valuable when complete data are not available or only extreme observations have been recorded, providing a powerful tool for extracting meaningful information. Therefore, studying and modeling record data is not only theoretically interesting but also has important practical applications in various statistical fields.

2. Lower Record Density Function of the LE Distribution

Let $l_1, l_2, \dots, l_r$ be a sequence of lower record values from the $LE(\theta, \lambda)$ distribution. Then, as shown by Ahsanullah and Nevzorov (2025), the density function of l_r is given by

$$f_r(x) = \frac{H^{r-1}(x)}{\Gamma(r)} f(x) \tag{5}$$

where $H(x) = -\ln F(x)$. By substituting $F_X(x)$ and $f_X(x)$ from ***Eq. (1)*** and ***Eq.* (2)**, respectively, the lower record density function $f_r(x)$ for the Lindley-Exponential distribution is obtained as

$$f_r(x) \quad = \frac{\lambda\theta^2}{\Gamma(r)(\theta+1)}(1+\lambda x)e^{-r\lambda\theta x}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{i+1}\frac{1}{i+1}\binom{i+1}{j}e^{-i\lambda\theta x}\left(\frac{\lambda\theta x}{\theta+1}\right)^j\right)^{r-1}. \tag{6}$$

According to Ahsanullah & Nevzorov [7] , the joint pdf of the lower record values l_m and l_n with $m < n$ is

$$f_{m,n}(x,y) = \frac{f(y)}{\Gamma(m)\Gamma(n-m)}[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}\left[\ln\left(\frac{F(x)}{F(y)}\right)\right]^{n-m-1},$$

$$0 < y < x < \infty, \qquad m < n.$$

By substituting $F_X(x)$ and $f_X(x)$ from ***Eq.* (1)**and ***Eq.* (2)**, respectively, the joint pdf of the lower record values $f_{m,n}(x,y)$ for the Lindley-Exponential distribution is obtained as

$$f_{m,n}(x,y) = \frac{\lambda^2\theta^4(1+\lambda x)(1+\lambda y)e^{-\lambda\theta(x+y)}}{\Gamma(m)\Gamma(n-m)(\theta+1)} \frac{\left[-\ln\left(1-\left(1+\frac{\lambda\theta x}{\theta+1}\right)e^{-\lambda\theta x}\right)\right]^{m-1}}{(\theta+1)-(\theta+1+\lambda\theta x)e^{-\lambda\theta x}} \times \left[\ln\left(\frac{\theta+1-(\theta+1+\lambda\theta x)e^{-\lambda\theta x}}{\theta+1-(\theta+1+\lambda\theta y)e^{-\lambda\theta y}}\right)\right]^{n-m-1} \tag{7}$$

3. Moments of Record Values from the LE Distribution

Definition 1 *The k-th moment of the r-th lower record value l_r is defined as*

$$\mu_r^{(k)} = \int_0^\infty x^k f_r(x)dx \text{for} k = 0,1,2,\ldots$$

Therefore, the variance is given by

$$\text{var}(l_r) = \mu_k^{(2)} - (\mu_k^{(1)})^2.$$

The r-th moment of the lower record value l_r from the $LE(\theta,\lambda)$ distribution is given by

$$\mu_n^{(k)} = \frac{\lambda\theta^2}{\Gamma(r)(\theta+1)}\int_0^\infty x^k\big(H(x)\big)^{r-1}(1+\lambda x)e^{-\lambda\theta x}dx. \tag{8}$$

Definition 2 *If the joint moment of order (k,s) of the lower record values l_m and l_n is denoted by $\mu_{m,n}^{(k,s)}$ where $(k,s = 0,1,2,\ldots)$, it is defined as follows*

$$\mu_{m,n}^{(k,s)} = \int_0^\infty \int_0^x x^k y^s f_{m,n}(x,y)dydx$$

Therefore, the covariance between records l_m and l_n is given by $cov(l_m,l_n) = \mu_{m,n} - \mu_m\mu_n$. Some numerical values of the mean of the lower record values from the Lindley-Exponential distribution for different values of r, θ, and λ are presented in ***Table 1***. It can be observed that the mean of the lower records decreases as r, θ, or λ increases.

Table 2 presents the values of the variances and covariances of the lower record statistics. It can be observed that as r increases, the variance and covariance of the record values decreases .However, with increasing values of θ and λ, the variance and covariance of the record statistics decreases.

Table 1. Mean of lower record statistics from the LE distribution

			$\lambda = 0.5$				
r	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$
1	6.67	3.00	1.87	1.33	1.03	0.83	0.70
2	2.79	1.18	0.71	0.50	0.31	0.30	0.25
3	1.34	0.54	0.32	0.22	0.17	0.13	0.11
4	0.67	0.26	0.15	0.10	0.08	0.06	0.05
5	0.34	0.13	0.07	0.05	0.04	0.03	0.02
			$\lambda = 1$				
1	3.33	1.5	0.93	0.67	0.51	0.42	0.35
2	1.39	0.59	0.35	0.25	0.19	0.15	0.13
3	0.67	0.27	0.16	0.11	0.09	0.07	0.06
4	0.34	0.13	0.08	0.05	0.04	0.03	0.03
5	0.17	0.06	0.04	0.03	0.02	0.01	0.01

Table 2.Variance and covariance of lower record statistics from the LE distribution

			$\lambda = 0.5$			
m	n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$
1	1	30.22	7.00	2.92	1.56	0.95
1	2	9.04	1.97	0.79	0.41	0.25
1	3	3.94	0.81	0.31	0.16	0.10
1	4	1.91	0.37	0.14	0.07	0.04
1	5	0.96	0.18	0.07	0.03	0.02
2	2	7.58	1.59	0.62	0.32	0.19
2	3	3.25	0.64	0.24	0.12	0.07
2	4	1.57	0.29	0.11	0.05	0.03
2	5	0.79	0.14	0.05	0.04	0.01
3	3	2.58	0.49	0.18	0.09	0.05
3	4	1.24	0.22	0.08	0.04	0.02
3	5	0.62	0.11	0.04	0.02	0.01
4	4	0.94	0.16	0.06	0.03	0.02
4	5	0.47	0.78	0.03	0.01	0.01
5	5	0.35	0.06	0.02	0.01	0.01
			$\lambda = 1$			
1	1	7.56	1.75	0.73	0.39	0.24
1	2	2.26	0.49	0.20	0.10	0.06
1	3	0.98	0.20	0.08	0.04	0.02
1	4	0.48	0.09	0.04	0.02	0.01
1	5	0.24	0.04	0.02	0.01	0.01
2	2	1.89	0.40	0.16	0.08	0.05
2	3	0.81	0.16	0.06	0.03	0.02
2	4	0.39	0.07	0.03	0.01	0.01
2	5	0.20	0.04	0.01	0.01	0.01
3	3	0.64	0.12	0.05	0.03	0.01
3	4	0.31	0.06	0.02	0.01	0.01
3	5	0.15	0.03	0.01	0.01	0.01
4	4	0.24	0.04	0.01	0.01	0.01
4	5	0.18	0.02	0.01	0.01	0.01
5	5	0.09	0.01	0.01	0.01	0.01

4. Maximum Likelihood Estimation

In this section, maximum likelihood (ML) estimators for the unknown parameters θ and λ of the LE distribution are derived using lower record values. To generate the record data, an inverse sampling procedure is employed, where observations are collected sequentially until the r-th lower record appears. Subsequently, ML estimates of θ and λ are obtained from these record values, and asymptotic confidence intervals (ACIs) for the parameters are constructed.

Suppose that $l_1, l_2, \dots, l_r$ is a sequence of lower record values from the $LE(\theta,\lambda)$ distribution. Then, the likelihood function for θ and λ based on the lower record values is given as follows (see Arnold et al . [9])):

$$\begin{aligned} L(\theta,\lambda;\boldsymbol{l}) &= f(l_r)\prod_{i=1}^{r-1}\frac{f(l_i)}{F(l_i)} \\ &= \frac{\lambda\theta^2}{\theta+1}(1+\lambda l_r)\ e^{-\lambda\theta l_r}\prod_{i=1}^{r-1}\frac{\frac{\lambda\theta^2}{\theta+1}(1+\lambda l_i)\ e^{-\lambda\theta l_i}}{1-\left(1+\frac{\lambda\theta l_i}{\theta+1}\right)e^{-\lambda\theta l_i}} \\ &= \lambda^r\left(\frac{\theta^2}{\theta+1}\right)^r\left[1-\left(1+\frac{\lambda\theta l_r}{\theta+1}\right)e^{-\lambda\theta l_r}\right]\prod_{i=1}^{r}\frac{(1+\lambda l_i)\ e^{-\lambda\theta l_i}}{1-\left(1+\frac{\lambda\theta l_i}{\theta+1}\right)e^{-\lambda\theta l_i}} \end{aligned} \tag{9}$$

The log-likelihood function is then:

$$\begin{aligned} L &= lnL(\theta,\lambda;\boldsymbol{l}) \\ &= rln\lambda + 2rln\theta - rln(\theta+1) - \lambda\theta\sum_{i=1}^{r} l_i \\ &+\sum_{i=1}^{r} ln(1+\lambda l_i) - \sum_{i=1}^{r-1} ln\left[1-\left(1+\frac{\lambda\theta l_i}{\theta+1}\right)e^{-\lambda\theta l_i}\right] \end{aligned} \tag{10}$$

Taking the partial derivatives of the log-likelihood function with respect to θ and λ and setting them equal to zero, we obtain the following likelihood equations:

$$\frac{\partial L}{\partial\theta} = \frac{2r}{\theta} - \frac{r}{\theta+1} - \lambda\sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1}\frac{-\lambda\theta l_i\big((\theta+2)+\lambda l_i(\theta+1)\big)e^{-\lambda\theta l_i}}{(\theta+1)^2\big(1-\big(1+(\lambda\theta l_i)/(\theta+1)\big)e^{-\lambda\theta l_i}\big)} = 0 \tag{11}$$

$$\frac{\partial L}{\partial\lambda} = \frac{r}{\lambda} - \theta\sum_{i=1}^{r} l_i + \sum_{i=1}^{r}\frac{l_i}{1+\lambda l_i} - \sum_{i=1}^{r-1}\frac{\theta^2 l_i(1+\lambda l_i)e^{-\lambda\theta l_i}}{\theta+1-(\theta+1+\lambda\theta l_i)e^{-\lambda\theta l_i}} = 0 \tag{12}$$

The estimates of the parameters θ and λ are denoted by $\hat{\theta}$ and $\hat{\lambda}$, respectively. Since the likelihood equations given in Eqs. (11) and (12) do not admit closed-form solutions, they must be solved numerically using simulated or real data. This numerical optimization is performed using the nlm package in the R software environment.

Furthermore, because the maximum likelihood estimators for θ and λ lack explicit expressions, obtaining exact confidence intervals is not feasible. Therefore, we rely on large sample approximations. According to the asymptotic theory of maximum likelihood estimation (e.g., Lawless, 1982), the distribution of the MLE $\hat{\boldsymbol{\theta}}$ is approximately normal:

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right) \sim N\left(0, I^{-1}(\boldsymbol{\theta})\right),$$

where $\boldsymbol{\theta}=(\theta,\lambda)^{\top}$ and $I^{-1}(\boldsymbol{\theta})$ represents the inverse of the observed Fisher information matrix. The observed information matrix is given by:

$$I(\theta,\lambda) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\theta^2}\right) & E\left(\frac{\partial^2 L}{\partial\theta\partial\lambda}\right) \\ E\left(\frac{\partial^2 L}{\partial\lambda\partial\theta}\right) & E\left(\frac{\partial^2 L}{\partial\lambda^2}\right) \end{bmatrix} \quad \textbf{(13)}$$

Since the expected values in ***Eq. (13)*** are difficult to compute, the variance-covariance matrix of the maximum likelihood estimators is instead approximated using the observed Fisher information matrix evaluated at the parameter estimates:

$$\hat{\Sigma} = -\begin{bmatrix} \frac{\partial^2 L}{\partial\theta^2} & \frac{\partial^2 L}{\partial\theta\partial\lambda} \\ \frac{\partial^2 L}{\partial\lambda\partial\theta} & \frac{\partial^2 L}{\partial\lambda^2} \end{bmatrix}^{-1}_{(\theta=\hat{\theta},\lambda=\hat{\lambda})} = \begin{bmatrix} var(\hat{\theta}) & cov(\hat{\theta},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\theta}) & var(\hat{\lambda}) \end{bmatrix}. \quad \textbf{(14)}$$

where

$$\frac{\partial^2 L}{\partial\theta^2} = -\frac{2r}{\theta^2} + \frac{r}{(\theta+1)^2} - \sum_{i=1}^{r-1} \frac{\left(-\lambda^2 l_i^2\left(1+\frac{\lambda\theta l_i}{\theta+1}\right) - \frac{2\lambda l_i}{(\theta+1)^3}\right)}{(F(l_i))^2} + \frac{\left(-\lambda l_i S(l_i) - \frac{\lambda}{\theta+1}e^{-\lambda\theta l_i}\right)^2}{(F(l_i))^2}$$

$$\frac{\partial^2 L}{\partial\lambda^2} = -\frac{r}{\lambda^2} - \sum_{i=1}^{r} \frac{l_i^2}{(1+\lambda l_i)^2} - \frac{\theta^2}{(\theta+1)^3} \sum_{i=1}^{r-1} \frac{l_i^2 e^{-\lambda\theta l_i}\left((\theta+1)+\lambda l_i\right)}{(F(l_i))^2}$$

$$\frac{\partial^2 L}{\partial\lambda\partial\theta} = -\sum_{i=1}^{r} l_i + \frac{l_i(1+\lambda l_i)}{(\theta+1)^2} \sum_{i=1}^{r-1} \frac{\theta e^{-\lambda\theta l_i}\left[(2-\lambda\theta l_i)(\theta+1)F(l_i) - \theta\left(1-\left(1+\lambda l_i(\theta+2+\lambda\theta l_i)e^{-\lambda\theta l_i}\right)\right)\right]}{(F(l_i))^2}$$

Let $\hat{\theta}$ and $\hat{\lambda}$ denote the ML estimators of θ and λ based on lower records. Then, the $100(1-\alpha)\%$ asymptotic confidence intervals (CIs) for θ and λ are given by

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{var(\hat{\theta})}$$

and

$$\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\lambda})}$$

where $var(\hat{\theta}) = -\frac{1}{n}\frac{\partial^2 L}{\partial \theta^2}|_{\theta=\hat{\theta}}$, $var(\hat{\lambda}) = -\frac{1}{n}\frac{\partial^2 L}{\partial \lambda^2}|_{\lambda=\hat{\lambda}}$ and $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$-th quantile of the standard normal distribution.

These confidence intervals may result in negative lower bounds. To overcome this issue, we apply a logarithmic transformation and, by using the delta method, obtain the asymptotic normal distributions of $ln(\theta)$ and $ln(\lambda)$ respectively, as

$$\sqrt{n}\left(ln\hat{\theta} - ln\theta\right) \sim N\left(0, \frac{var\hat{(}\theta)}{\theta^2}\right),$$

$$\sqrt{n}\left(ln\hat{\lambda} - ln\lambda\right) \sim N\left(0, \frac{var\hat{(}\lambda)}{\lambda^2}\right)$$

Now, the asymptotic $100(1-\alpha)\%$ confidence intervals of lnθ and lnλ, respectively, are

$$ln\hat{\theta} \pm Z_{\frac{\alpha}{2}}\frac{\sqrt{var(\hat{\theta})}}{\hat{\theta}} \equiv (L_1, U_1)$$

and

$$ln\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\frac{\sqrt{var(\hat{\lambda})}}{\hat{\lambda}} \equiv (L_2, U_2).$$

Finally, using the inverse logarithmic transformation, the asymptotic $100(1-\alpha)\%$ confidence intervals of θ and λ, respectively, can be obtained as (e^{L_1}, e^{U_1}), (e^{L_2}, e^{U_2}).

5. Bayesian Estimation

In this section, we develop Bayesian inference procedures for the parameters of the Lindley–Exponential (LE) distribution. Record values are inherently rare. Therefore, incorporating prior information is both useful and desirable, making Bayesian methods particularly suitable for analyzing such data.

For Bayesian estimation, two loss functions are considered: the symmetric squared error (SE) loss and the asymmetric linear–exponential (LINEX) loss function Varian [24]. The squared error loss function is defined as$L_{SE}(g(\theta), g(\hat{\theta})) = [g(\theta) - g(\hat{\theta})]^2$ where $g(\hat{\theta})$denotes an estimate of the function $g(\theta)$. Accordingly, the Bayes estimate of $g(\theta)$under the SE loss, denoted by $\hat{g}_{SE}(\theta)$, is the posterior mean.

The LINEX loss function, which accounts for estimation asymmetry, is expressed as

$$L_{LE}\left(g(\theta), g(\hat{\theta})\right) = \exp\left[v\left(g(\theta) - g(\hat{\theta})\right)\right] - v\left(g(\theta) - g(\hat{\theta})\right) - 1, v \neq 0.$$

Positive values of $v$imply that overestimation is penalized more heavily than underestimation, while negative values have the opposite effect. As $v \to 0$, the LINEX loss approximates the SE loss and thus behaves nearly symmetrically.

Under the LINEX loss function, the Bayes estimate of $g(\theta)$ is expressed as

$$\hat{g}_{LE}(\theta) = -\frac{1}{v}\log(E_\theta[\exp(-vg(\theta))|\delta]).$$

Since the parameters of the LE distribution are positive, we assume independent gamma prior distributions for them. Specifically, $\theta$and $\lambda$are assumed to follow gamma priors with probability density functions

$$\pi(\theta) \propto \theta^{a-1}e^{-b\theta}, \pi(\lambda) \propto \lambda^{c-1}e^{-d\lambda}, \quad \textbf{(15)}$$

where $a, b, c,$and $d$are known hyperparameters taking nonnegative values.

5.1. Bayesian Estimation Based on Record Values

Combining ***Eq. (9)*** and ***(15)*** and applying Bayes' theorem, the joint posterior distribution of the parameters $\theta$and λgiven the observed lower record values $\boldsymbol{l} = (l_1, \dots, l_r)$is proportional to

$$\pi(\theta, \lambda|\boldsymbol{l}) \propto \lambda^{r+c-1}\frac{\theta^{2r+a-1}}{(\theta+1)^r} exp\left\{-\left(b + \lambda\sum_{i=1}^{r} l_i\right)\theta - d\lambda\right\}\prod_{i=1}^{r-1}\frac{1+\lambda l_i}{\theta + 1 - (\theta + 1 + \lambda\theta l_i)e^{-\lambda\theta l_i}}. \quad \textbf{(16)}$$

Hence, the normalized joint posterior distribution is given by

$$\pi^*(\theta, \lambda|\boldsymbol{l}) = \frac{1}{D}\pi(\theta, \lambda; \boldsymbol{l}), \quad \textbf{(17)}$$

where

$$D = \int_0^\infty \int_0^\infty \pi(\theta, \lambda; \mathbf{l}) d\theta d\lambda$$

is the normalizing constant.

Based on the SE and LINEX loss functions, the Bayes estimators of a function $g(\theta, \lambda)$are obtained as follows Under the SE loss function, the Bayes estimator is

$$\hat{g}_{SE} = E[g(\theta,\lambda)|\boldsymbol{l}] = \frac{1}{D}\int_0^\infty \int_0^\infty g(\theta,\lambda)\pi(\theta,\lambda;\boldsymbol{l})d\theta d\lambda. \quad (18)$$

Under the LINEX loss function, the corresponding Bayes estimator is given by

$$\begin{aligned}\hat{g}_{LE} &= -\frac{1}{v}log(E[exp\{-vg(\theta,\lambda)\}|\boldsymbol{l}]) \\ &= -\frac{1}{v}log\left(\frac{1}{D}\int_0^\infty \int_0^\infty exp - vg(\theta,\lambda)\pi(\theta,\lambda;\boldsymbol{l})d\theta d\lambda\right).\end{aligned} \quad (19)$$

Since the Bayes estimators involve double integrals that do not have closed-form solutions, numerical methods are employed to obtain approximate estimates. Specifically, the Laplace approximation proposed by Tierney and Kadane [25] is adopted, and the Metropolis–Hastings (M–H) algorithm is implemented within the Gibbs sampling framework to generate random samples from the posterior distribution.

I. Tierney and Kadane’s Approximation

Let $\varphi(\theta,\lambda) = \frac{1}{n}\ln\pi(\theta,\lambda;\boldsymbol{l})$ and $\varphi^*(\theta,\lambda) = \varphi(\theta,\lambda) + \frac{1}{n}\ln g(\theta,\lambda)$. ***Eq. (18)*** can be written as

$$\hat{g}_{SE} = E(g(\gamma,\delta)|\boldsymbol{l}) = \frac{\int_0^\infty \int_0^\infty e^{n\ \varphi^*(\gamma,\delta)}d\theta d\lambda}{\int_0^\infty \int_0^\infty e^{n\varphi(\gamma,\delta)}d\theta d\lambda} \quad (20)$$

Using the Tierney and Kadane method, it will have the following approximate form

$$\hat{g}_{BT}(\theta,\lambda) = \frac{\int_0^\infty \int_0^\infty e^{n\varphi^*(\theta,\lambda)}d\theta d\lambda}{\int_0^\infty \int_0^\infty e^{n\varphi(\theta,\lambda)}d\theta d\lambda} \approx \sqrt{\frac{det\psi^*}{det\psi}}exp\{n[\varphi^*(\bar{\theta}^*,\bar{\lambda}^*) - \varphi(\bar{\theta},\bar{\lambda})]\}, \quad (21)$$

where $(\bar{\theta}^*,\bar{\lambda}^*)$ and $(\bar{\theta},\bar{\lambda})$ denote the values that maximize φ^* and φ, respectively, while ψ^* and ψ represent the negatives of the inverse Hessian matrices of φ^* and φ evaluated at their respective maxima. The general form of $\varphi(\theta,\lambda)$ is

$$\begin{aligned}\varphi(\theta,\lambda) = \frac{1}{n}k &+ (r+c-1)ln\lambda + (2r+a-1)ln\theta - rln(\theta+1) \\ &- \left(b + \lambda\sum_{i=1}^{r} l_i\right)\theta - d\lambda + \sum_{i=1}^{r} ln(1+\lambda l_i) \\ &- \sum_{i=1}^{r-1} ln\left[1 - \left(1 + \frac{\lambda\theta l_i}{\theta+1}exp(-\lambda\theta l_i)\right)\right],\end{aligned} \quad (22)$$

with k being independent of the parameters.

The equations to determine $(\bar{\theta},\bar{\lambda})$ are given by setting the partial derivatives of φ to zero

$$\frac{\partial\varphi}{\partial\theta}=\frac{2r+a-1}{\theta}-\frac{r}{\theta+1}-\left(b+\lambda\sum_{i=1}^{r}l_i\right)-\sum_{i=1}^{r-1}\frac{\lambda l_i e^{-\lambda\theta l_i}}{(\theta+1)(\theta+1-\lambda\theta l_i e^{-\lambda\theta l_i})}=0,$$

$$\frac{\partial\varphi}{\partial\lambda}=\frac{r+c-1}{\lambda}-\theta\sum_{i=1}^{r}l_i-d+\sum_{i=1}^{r}\frac{l_i}{1+\lambda l_i}-\sum_{i=1}^{r-1}\frac{\theta l_i e^{-\lambda\theta l_i}(1-\lambda\theta l_i)}{\theta+1-\lambda\theta l_i e^{-\lambda\theta l_i}}=0.$$

and $\psi=(\varphi_{11}\varphi_{22}-\varphi_{12})^{-1}$ where

$$\varphi_{11}=\frac{\partial^2\varphi}{\partial\theta^2}=\frac{1}{n}[-\frac{2r+a-1}{\theta^2}+\frac{r}{(\theta+1)^2}-\sum_{i=1}^{r-1}\frac{-\lambda^2 l_i^2 e^{-\lambda\theta l_i}\cdot D_i-\lambda l_i e^{-\lambda\theta l_i}\cdot D_i'}{D_i^2}]_{\theta=\bar{\theta},\lambda=\bar{\lambda}}$$

such that

$$D_i=(\theta+1)^2\left(1-\frac{\lambda\theta l_i}{\theta+1}e^{-\lambda\theta l_i}\right)$$

$$D_i'=2(\theta+1)\left(1-\frac{\lambda\theta l_i}{\theta+1}e^{-\lambda\theta l_i}\right)+(\theta+1)^2\left[-\frac{\lambda l_i}{(\theta+1)^2}e^{-\lambda\theta l_i}+\frac{\lambda^2\theta l_i^2}{\theta+1}e^{-\lambda\theta l_i}\right]$$

and

$$\varphi_{22}=\frac{\partial^2\varphi}{\partial\lambda^2}=\frac{1}{n}[-\frac{r+c-1}{\lambda^2}-\sum_{i=1}^{r}\frac{l_i^2}{(1+\lambda l_i)^2}-\sum_{i=1}^{r-1}\frac{C_i'F_i-C_iF_i'}{F_i^2}]_{\theta=\bar{\theta},\lambda=\bar{\lambda}}$$

such that

$$C_i=\frac{\theta l_i}{\theta+1}e^{-\lambda\theta l_i}(1-\lambda\theta l_i)$$

$$C_i'=\frac{\theta l_i}{\theta+1}e^{-\lambda\theta l_i}\cdot[-\theta l_i(2-\lambda\theta l_i)]$$

$$F_i=1-\frac{\lambda\theta l_i}{\theta+1}e^{-\lambda\theta l_i}$$

$$F_i'=-\frac{\theta l_i}{\theta+1}e^{-\lambda\theta l_i}(1-\lambda\theta l_i)$$

and

$$\varphi_{12}=\frac{\partial^2\varphi(\theta,\lambda)}{\partial\theta\partial\lambda}=-\frac{1}{n}\sum_{i=1}^{r-1}\left[\frac{G_i'\cdot F_i-G_i\cdot F_i'}{D_i^2}\right]_{\theta=\bar{\theta},\lambda=\bar{\lambda}}$$

such that

$$G_i=\frac{\theta l_i e^{-\lambda\theta l_i}}{\theta+1}(1-\lambda\theta l_i)$$

$$G_i'=\left[\frac{l_i e^{-\lambda\theta l_i}}{\theta+1}-\frac{\theta l_i e^{-\lambda\theta l_i}}{(\theta+1)^2}-\frac{\lambda l_i^2 e^{-\lambda\theta l_i}}{\theta+1}+\frac{\lambda\theta^2 l_i^3 e^{-\lambda\theta l_i}}{\theta+1}\right]$$

Now, to calculate the Bayesian estimate of θ based on the SE error loss, let $g(\theta,\lambda)=\theta$. Hence,$(\bar{\theta}^*,\ \bar{\lambda}^*)$ can be calculated by maximizing the following relation: Similarly, for $g(\theta,\lambda)=\theta$ under the squared error loss, we redefine

$$\varphi^*(\theta,\lambda)=\varphi(\theta,\lambda)+\frac{1}{n}\ln\theta,$$

and the equations to determine $(\bar{\theta}^*,\bar{\lambda}^*)$ become:

$$\frac{\partial\varphi^*}{\partial\theta}=\frac{2r+a}{\theta}-\frac{r}{\theta+1}-\left(b+\lambda\sum_{i=1}^{r}l_i\right)-\sum_{i=1}^{r-1}\frac{\lambda l_i e^{-\lambda\theta l_i}}{(\theta+1)(\theta+1-\lambda\theta l_i e^{-\lambda\theta l_i})}=0=0,$$

$$\frac{\partial\varphi^*}{\partial\lambda}=\frac{r+c-1}{\lambda}-\theta\sum_{i=1}^{r}l_i-d+\sum_{i=1}^{r}\frac{l_i}{1+\lambda l_i}-\sum_{i=1}^{r-1}\frac{\theta l_i e^{-\lambda\theta l_i}(1-\lambda\theta l_i)}{\theta+1-\lambda\theta l_i e^{-\lambda\theta l_i}}=0..$$

and $\psi^*=(\varphi^*_{11}\varphi^*_{22}-\varphi^*_{12})^{-1}$ where

$$\varphi^*_{11}=\frac{1}{n}[-\frac{2r+a}{\theta^2}+\frac{r}{(\theta+1)^2}-\sum_{i=1}^{r-1}\frac{-\lambda^2 l_i^2 e^{-\lambda\theta l_i}\cdot D_i-\lambda l_i e^{-\lambda\theta l_i}\cdot D_{i'}}{D_i^2}]_{\theta=\bar{\theta},\lambda=\ \bar{\lambda}}$$

such that

and $=\varphi_{22}|_{\theta=\bar{\theta}^*,\lambda=\bar{\lambda}^*}$, $\varphi^*_{12}=\varphi_{12}|_{\theta=\bar{\theta}^*,\lambda=\bar{\lambda}^*}$. Then, an approximation of the Bayesian estimator of θ under the SE loss is given by

$$\hat{\theta}_{SET}(\theta,\lambda)\approx\sqrt{\frac{det\psi^*}{det\psi}}exp\{n[\varphi^*(\bar{\theta}^*,\bar{\lambda}^*)-\varphi(\bar{\theta},\bar{\lambda})]\}. \quad \textbf{(23)}$$

For LINEX loss function with $g(\theta,\lambda)=e^{-v\theta}$, we define $\varphi^{**}=\varphi+\frac{1}{n}\ln e^{-v\theta}=\varphi-\frac{v}{n}\theta$, and maximize it accordingly.

Following the same procedure for $g(\theta,\lambda)=\lambda$ and $g(\theta,\lambda)=e^{-v\lambda}$, the approximate Bayes estimators of λ under SE and LE loss functions can also be derived.

II. Metropolis-Hastings within Gibbs Sampling

When employing the Laplace approximation for Bayesian parameter estimation, it is not possible to derive highest posterior density (HPD) credible intervals. To address this, we apply the Metropolis–Hastings algorithm within a Gibbs sampling framework to generate samples from the conditional posterior distributions of the parameters. These samples are subsequently used to calculate both the HPD credible intervals and the point Bayes estimates.

Using ***Eq. (16)***, the conditional posterior density functions of θ and λ are given by

$$\pi_1^*(\theta|\lambda,\boldsymbol{l}) \propto \frac{\theta^{2r+a-1}}{(\theta+1)^r} exp\left\{-\left(b+\lambda\sum_{i=1}^{r} l_i\right)\theta - d\lambda\right\} \times \prod_{i=1}^{r-1} \frac{1+\lambda l_i}{\theta+1-(\theta+1+\lambda\theta l_i)e^{-\lambda\theta l_i}} \tag{24}$$

$$\pi_2^*(\lambda|\theta,\boldsymbol{l}) \propto \lambda^{r+c-1} exp\left\{-\left(b+\lambda\sum_{i=1}^{r} l_i\right)\theta - d\lambda\right\} \times \prod_{i=1}^{r-1} \frac{1+\lambda l_i}{\theta+1-(\theta+1+\lambda\theta l_i)e^{-\lambda\theta l_i}}. \tag{25}$$

Since these conditional distributions are not of standard form, direct sampling is not feasible. Therefore, we employ the Metropolis–Hastings (M–H) algorithm using normal proposal distributions to generate the samples.

Algorithm 1: Metropolis–Hastings embedded in a Gibbs sampling framework

a) Set initial values: $\left(\theta^{(0)},\lambda^{(0)}\right) = \left(\hat{\theta}_{ML},\hat{\lambda}_{ML}\right)$ and set $j=1$.

b) Generate $\theta^{(j)}$ from $\pi_1^*\left(\theta\middle|\lambda^{(j-1)},\boldsymbol{l}\right)$ utilizing the M–H method based on a proposal distribution $q(\theta) \sim N\left(\theta^{(j-1)},\sigma_1^2\right)$
- Set $\nu_1 = \theta^{(j-1)}$.
- Generate $w_1 \sim N(\nu_1,1)$.
- Compute acceptance probability

$$P(\nu_1,w_1) = \min\left\{1,\frac{\pi_1^*(w_1)q(\nu_1)}{\pi_1^*(\nu_1)q(w_1)}\right\}.$$

- Generate $\alpha \sim U(0,1)$. If $P(\nu_1,w_1) \geq \alpha$, set $\theta^{(j)} = w_1$; else, $\theta^{(j)} = \nu_1$.

c) Generate $\lambda^{(j)}$ from $\pi_2^*\left(\lambda\middle|\theta^{(j)},\boldsymbol{l}\right)$ utilizing the M–H method based on a proposal distribution $q(\lambda) \sim N\left(\lambda^{(j-1)},\sigma_2^2\right)$:
- Set $\nu_2 = \lambda^{(j-1)}$.
- Generate $w_2 \sim N(\nu_2,1)$.
- Compute acceptance probability

$$P(\nu_2,w_2) = \min\left\{1,\frac{\pi_2^*(w_2)q(\nu_2)}{\pi_2^*(\nu_2)q(w_2)}\right\}.$$

- Generate $\alpha \sim U(0,1)$. If $P(\nu_2,w_2) \geq \alpha$, set $\lambda^{(j)} = w_2$; else, $\lambda^{(j)} = \nu_2$.

d) Increment j by 1 and continue Steps 2–4 for a total of M iterations to generate samples $\theta^{(j)}$ and $\lambda^{(j)}$ for $j = 1,2,\dots,M$.

Using the generated samples, the Bayesian estimates of the parameters with respect to the squared error loss are given by

$$\tilde{\theta}_{SEM} = \frac{1}{M}\sum_{j=1}^{M} \theta^{(j)}, \qquad \tilde{\lambda}_{SEM} = \frac{1}{M}\sum_{j=1}^{M} \lambda^{(j)}. \tag{26}$$

Using the LINEX (LE) loss, the Bayesian estimates of the parameters are given by

$$\tilde{\theta}_{LEM} = -\frac{1}{v} log\left(\frac{1}{M}\sum_{j=1}^{M} e^{-v\theta^{(j)}}\right), and \tilde{\lambda}_{LEM} = -\frac{1}{v} log\left(\frac{1}{M}\sum_{j=1}^{M} e^{-v\lambda^{(j)}}\right). \tag{27}$$

Let $\theta^{(1)} < \theta^{(2)} < \cdots < \theta^{(M)}$ denote the ordered samples of θ. Then, the $100(1-\alpha)\%$ highest posterior density (HPD) credible interval for θ is defined as the shortest interval among the sequences $\left(\theta^{(1)}, \theta^{((1-\alpha)M)}\right), \dots, \left(\theta^{(\alpha M)}, \theta^{(M)}\right)$.

A similar approach is used to determine the HPD credible interval for λ.

6. Forecasting Future Record Values

Forecasting future record values has attracted considerable attention in various disciplines, including seismology, sports, economics, agriculture, meteorology, industrial processes, and other scientific areas. In this section, we focus on predicting future lower record values. Initially, we provide non-Bayesian point predictors using the observed lower records. Subsequently, we present Bayesian point and interval estimates for upcoming lower record observations. Specifically, we derive point estimates and associated prediction intervals for the mth lower record value, where $m > r$ and r is the number of observed lower records.

6.1. ML-Based Forecasting of Subsequent Record Values

Suppose that $l_1, l_2, \dots, l_r$ is a sequence of lower record values from a distribution with cumulative distribution function $F(.;\theta)$ and probability density function $f(.;\theta)$ where θ denotes the vector of unknown parameters. Basak & Balakrishnan [26] derived the predictive likelihood function for θ and the future record $y = l_m$, $(m > r)$ as

$$L(y, \boldsymbol{\theta}; \boldsymbol{l}) = \frac{[H(y, \boldsymbol{\theta}) - H(l_r, \boldsymbol{\theta})]^{m-r-1}}{\Gamma(m-r)} f(y, \boldsymbol{\theta}) \prod_{i=1}^{r} \frac{f(l_i)}{F(l_i)} \tag{28}$$

where $\boldsymbol{\theta} = (\theta, \lambda)$, $\boldsymbol{l} = (l_1, l_2, \dots, l_r)$.

Using ***Eq. (1)***, ***Eq. (2),*** and ***Eq. (28)***, we can express the logarithm of the predictive likelihood function (ignoring additive constants) as follows

$$\ell(y,\theta,\lambda;\boldsymbol{l}) = (m-r-1)ln[H(y,\theta,\lambda)-H(l_r,\theta,\lambda)] + 2(r+1)ln\theta + (r+1)ln\lambda - \lambda\theta y - (r+1)ln(\theta+1) + ln(1+\lambda y) + \sum_{i=1}^{r} ln(1+\lambda l_i) - \sum_{i=1}^{r} \lambda\theta l_i + \sum_{i=1}^{r} H(l_i,\theta,\lambda). \tag{29}$$

The parameters in question can be estimated by maximizing the predictive log-likelihood (29); this produces the MLP for $y = l_m$ and the PMLEs of θ and λ using the lower-record data.

6.2. Bayesian Forecasting of Upcoming Record Values

We intend to predict the $y = l_m, (m > r)$ under the Bayesian method. To predict future records, we first derive the conditional probability density function of $y = l_m$ given the r observed lower records:

$$f(y|\theta,\lambda,\boldsymbol{l}) = \frac{\lambda\theta^2(1+\lambda y)e^{-\lambda\theta y}}{(\theta+1)\Gamma(m-r)}\left[1 - \left(1+\frac{\lambda\theta l_r}{\theta+1}\right)e^{-\lambda\theta l_r}\right]^{-1}\left\{ln\left(\frac{(\theta+1)-(\theta+1+\lambda\theta l_r)e^{-\lambda\theta l_r}}{(\theta+1)-(\theta+1+\lambda\theta l_r)e^{-\lambda\theta y}}\right)\right\}^{m-r-1} \tag{30}$$

Using ***Eq. (30)***, the posterior predictive density and survival functions of y can then be expressed as

$$h(y|\boldsymbol{l}) = \int_0^\infty \int_0^\infty f(y|\theta,\lambda,\boldsymbol{l})\pi^*(\theta,\lambda|\boldsymbol{l})d\theta d\lambda \tag{31}$$

$$H(y|\boldsymbol{l}) = \int_0^\infty \int_0^\infty S^*(y|\theta,\lambda,\boldsymbol{l})\pi^*(\theta,\lambda|\boldsymbol{l})d\theta d\lambda \tag{32}$$

where

$$S^*(y|\theta,\lambda,\boldsymbol{l}) = \int_y^\infty f(u|\theta,\lambda,\boldsymbol{l})du \tag{33}$$

The integral in ***Eq. (33)*** does not admit a closed-form solution and therefore must be approximated numerically. To accomplish this, we utilize the Metropolis–Hastings's algorithm embedded within a Gibbs sampling framework.

Suppose $(\theta^j,\lambda^j), j = 1,\ldots,M$ are random samples generated from $\pi^*(\theta,\lambda|\boldsymbol{l})$ using the M–H technique based on a normal proposal mechanism. The steps of the algorithm are outlined below

1. Set initial values $(\theta^0,\lambda^0) = (\hat{\theta}_{ML},\hat{\lambda}_{ML})$ and $j = 1$.
2. Generate θ^j from $\pi^*(\theta|\lambda^{j-1},\boldsymbol{l})$ using proposal $q(\theta) \sim \mathcal{N}(\theta^{j-1},\sigma_1^2)$:(usually $\sigma^2 = 1$ and chosen so that the acceptance rate falls between 20% and 70%.)

- Set $\nu_1 = \theta^{j-1}$.
- Generate $w_1 \sim q(\theta)$.
- Compute acceptance ratio

$$P(\nu_1, w_1) = \min\left\{1, \frac{\pi^*(w_1)q(\nu_1)}{\pi^*(\nu_1)q(w_1)}\right\}$$

- Generate $\alpha \sim U(0,1)$. If $P(\nu_1, w_1) \geq \alpha$, set $\theta^j = w_1$, otherwise $\theta^j = \nu_1$.

3. Generate λ^j from $\pi^*(\lambda|\theta^j, \boldsymbol{l})$ using proposal $q(\lambda) \sim N(\lambda^{j-1}, \sigma_2^2)$:
   - Set $\nu_2 = \lambda^{j-1}$.
   - Generate $w_2 \sim q(\lambda)$.
   - Compute acceptance ratio

$$P(\nu_2, w_2) = \min\left\{1, \frac{\pi^*(w_2)q(\nu_2)}{\pi^*(\nu_2)q(w_2)}\right\}$$

   - Generate $\alpha \sim U(0,1)$. If $P(\nu_2, w_2) \geq \alpha$, set $\lambda^j = w_2$, otherwise $\lambda^j = \nu_2$.

Set $j = j + 1$ and repeat Steps 2-4 for M iterations.

Then, the posterior estimates of the predictive density and survival function are approximated as

$$\hat{h}(y|\boldsymbol{l}) \approx \frac{1}{M}\sum_{j=1}^{M} f(y|\theta^j, \lambda^j, \boldsymbol{l}) \quad \textbf{(34)}$$

$$\hat{H}(y|\boldsymbol{l}) \approx \frac{1}{M}\sum_{j=1}^{M} S^*(y|\theta^j, \lambda^j, \boldsymbol{l}) \quad \textbf{(35)}$$

The predictive values of $y = l_m$ under the SE and LE loss functions, derived from $\hat{h}(y|\boldsymbol{l}, \boldsymbol{t})$, are

$$\hat{y}_{SE} = \int_0^{l_r} y\hat{h}(y|\boldsymbol{l})dy = \frac{1}{M}\sum_{j=1}^{M}\int_0^{l_r} yf(y|\theta^j, \lambda^j, \boldsymbol{l})dy \quad \textbf{(36)}$$

$$\hat{y}_{LE} = -\frac{1}{v}ln\int_0^{l_r} e^{-vy}\hat{h}(y|\boldsymbol{l})dy = -\frac{1}{v}ln\left\{\frac{1}{M}\sum_{j=1}^{M}\int_0^{l_r} e^{-vy}f(y|\theta^j, \lambda^j, \boldsymbol{l})dy\right\} \quad \textbf{(37)}$$

The $100(1-\alpha)\%$ symmetric prediction interval (L_1, L_2) for $y = l_m$ is determined by solving

$$\hat{H}(L_1|\boldsymbol{l}) = 1 - \frac{\alpha}{2}, \quad \& \quad \hat{H}(L_2|\boldsymbol{l}) = \frac{\alpha}{2}. \quad \textbf{(38)}$$

These two integrals can also be approximated using the M-H method embedded in the Gibbs sampling procedure.

7. Numerical comparisons

7.1 Simulation Study

In this section, we perform Monte Carlo simulations to assess the effectiveness of the proposed methods across different sample sizes. The estimators are evaluated based on their estimated risks (ER), while point predictors are compared using the mean squared prediction errors (MSPEs). Furthermore, the performance of confidence and prediction intervals is examined in terms of both average lengths and coverage probabilities.

For the parameter θ, the estimated risks under SEL and LINEX losses are determined as follows

$$ER_{\text{SEL}}(\theta) = \frac{1}{N}\sum_{i=1}^{N} (\theta_i - \hat{\theta}_i)^2 \tag{39}$$

$$ER_{\text{LE}}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \left[\exp\left(v(\theta_i - \hat{\theta}_i)\right) - v(\theta_i - \hat{\theta}_i) - 1\right] \tag{40}$$

Where, $\hat{\theta}_i$ is the estimate of θ, and N denotes the total number of simulation replications.

Random samples are generated from the Lindley-Exponential (LE) distribution with parameters $(\theta,\lambda) = (2,3),(1,1)$, and the lower record values are sequentially extracted until the r-th lower record is observed ($r = 2,3,4,5,10$). Then, the MLEs of the parameters are calculated once using the likelihood function of ***Eq. (13)***.

The average values (AVs) and estimation risks (ERs) for the MLEs are computed based on 12000 repetitions are presented in ***Tables 3-6***. For the Bayesian analysis, we consider two distinct sets of hyperparameter values: $(a,b) = (2,1),(c,d) = (2,1)$ (Prior I) and $(a,b) = (4,2),(c,d) = (4,1))$ (Prior II), in order to examine the sensitivity of the predictors and estimators to different informative priors. The approximate Bayes estimates of the parameters under the LINEX loss function ($\nu = 0.5,0.5,1$) and the squared error (SE) loss function are obtained using the methods of Tierney and Kadane as well as Gibbs sampling.

For the Gibbs procedure, Markov chains of length 90,000 are generated, with the first 30,000 samples discarded to mitigate the effect of the initial values. To reduce autocorrelation among the retained samples, every 5th observation is selected, yielding final chains of size 12,000. Convergence of the MCMC samples is assessed using the approach of Gelman [28] where the potential scale reduction factor is calculated as $\sqrt{Var(\Delta)/W}$, with Δ representing the estimate of interest and $Var(\Delta) = (n-1)W/n + Z/n$ where n is the number of iterations per chain, and W and Z are the within-chain and between-chain variances, respectively. Scale reduction factors below 1.1 indicate satisfactory convergence of the chains.

The plots of the simulated values of θ and λ are given in ***Figure 5-Figure 8*** which shows the convergence of Gibbs algorithm. Additionally, the results of the Gelman–Rubin convergence diagnostic ($\hat{R}$) for the parameters $\theta$and λat different values of $r$under Prior I and Prior II respectively are presented in ***Tables 8*** and ***9***. In all cases, the values of $\hat{R}$are less than 1.1, indicating satisfactory convergence of the MCMC process. The plots and tables are presented in Appendix A

1. As expected, with increasing sample size, the maximum likelihood estimates tend to approach the true parameter values.
2. The Bayesian estimates of the parameters θ and λ under the squared error (SE) loss function generally outperform the maximum likelihood estimates (MLEs) in terms of lower expected risk (ER). However, direct comparison between the Bayesian estimates under the LINEX loss function and the MLEs is not straightforward. In addition, it is observed that, for some scenarios, the ERs of the Bayesian estimates obtained via the MCMC method are smaller than those derived using the Tierney and Kadane approach, while in other scenarios the reverse holds. Nonetheless, as the sample size increases, the ER values from different methods converge and approach similar levels.
3. Next, the performance of the non-Bayesian predictors is summarized in Table 7. The Bayesian predictors, derived under both the LINEX and squared error (SE) loss functions, were also evaluated. To assess the sensitivity of the Bayesian predictors to the choice of hyperparameters, the two sets of priors described earlier were used. Table 3-
4. ***Table 6*** report the average estimates of the $(r+1)$th record values, along with their corresponding mean squared prediction errors (MSPEs) and 95% prediction intervals (PIs). It is observed that, in terms of MSPE, the point predictors based on Prior I perform slightly better than those based on Prior II. Furthermore, the prediction intervals obtained using Prior I tend to be shorter than those from Prior II. Overall, as the sample size increases, the predictive performances converge and become increasingly similar across methods.

Table 3. MLEs and Bayesian parameter estimates corresponding to Prior I with $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{2}, \mathbf{3})$.

r			MLE	Bayes estimates using MCMC method				Bayes estimates using T–K			
				SEL	Linex			SEL	Linex		
					$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$		$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$
2	θ	AV	2.788	1.2973	1.5444	1.1471	1.0403	1.9676	1.9796	1.5425	1.2437
		ER	1.506	0.7334	0.0949	0.1435	1.3115	0.0020	0.0002	0.0283	0.3740
	λ	AV	2.834	1.4302	1.7657	1.2350	1.1037	1.9831	1.9883	1.5519	1.2498
		ER	0.970	0.9689	0.1353	0.2061	2.9359	1.0343	0.1089	0.3387	3.005
3	θ	AV	2.704	0.8630	0.9616	0.7914	0.7359	1.9782	1.9893	1.5468	1.2458
		ER	1.587	0.3304	0.0396	0.0530	0.3280	5e-05	0.0009	0.0277	0.3717
	λ	AV	2.844	1.3981	1.7341	1.2085	1.0759	1.9887	1.9941	1.5532	1.2499
		ER	0.959	0.9520	0.1326	0.2058	3.7080	0.1077	1.0228	0.3380	3.0050
4	θ	AV	2.660	0.5727	0.6267	0.5339	0.5035	1.9836	1.9935	1.5490	1.2469
		ER	1.635	0.1794	0.0205	0.0281	0.1622	0.0005	2e-05	0.2750	0.3705
	λ	AV	2.843	1.3837	1.6940	1.1972	1.0680	1.9916	1.9965	1.5538	1.2450
		ER	0.960	0.9225	0.1268	0.1887	2.2663	1.0169	0.1072	0.3377	3.0048
5	θ	AV	2.613	0.4076	0.4315	0.3878	0.3710	1.9868	1.9555	1.5503	1.2475
		ER	1.710	0.0866	0.0101	0.0122	0.0575	0.0003	1e-05	0.2729	0.3698
	λ	AV	2.860	1.2939	1.5494	1.1290	1.0124	1.9930	1.9976	1.5542	1.2450
		ER	0.961	0.8016	0.1062	0.1515	1.3352	1.0135	0.1070	0.3375	3.0047
10	θ	AV	2.530	0.1351	0.1374	0.1330	0.1307	1.9934	1.9988	1.5530	1.2488
		ER	1.836	0.0088	0.0011	0.0011	0.0047	8e-05	1e-06	0.0270	0.3684
	λ	AV	2.872	1.5118	1.8263	1.3241	1.1937	1.9967	1.9994	1.5549	1.2450
		ER	0.9670	0.9298	0.1282	0.1916	2.3172	1.0066	0.1067	0.3371	3.0046

Table 4. Bayes estimators corresponding to Prior II with $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{2}, \mathbf{3})$.

r			Bayes estimates using MCMC method				Bayes estimates using T–K			
			SEL	Linex			SEL	Linex		
				$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$		$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$
2	θ	AV	1.1509	1.2446	1.0764	1.0150	1.9674	2.1478	1.7430	1.5457
		ER	0.3314	0.0409	0.0505	0.2876	0.0021	0.0034	0.0087	0.1209
	λ	AV	2.39.9	2.8913	2.0722	1.8521	3.9639	3.9951	3.0977	2.4950
		ER	1.5660	0.2559	0.3468	4.5162	0.9305	0.1472	0.0012	0.1520
3	θ	AV	0.8096	0.8634	0.7662	0.7301	1.9779	2.1712	1.7486	1.5489
		ER	0.1918	0.0229	0.0281	0.1451	1.0450	0.0753	0.2439	1.8167
	λ	AV	2.2579	2.7881	1.9551	1.7473	3.9763	3.9987	3.1024	2.4968
		ER	1.5124	0.2515	0.3652	8.1175	3.9065	0.7172	0.1275	0.1053
4	θ	AV	0.5719	0.6011	0.5972	0.5258	1.9835	2.1837	1.7514	1.5506
		ER	0.1069	0.0127	0.0149	0.0708	1.0335	0.0731	0.2465	1.8111
	λ	AV	2.0657	2.5880	1.7892	1.6015	3.9825	3.9997	3.1048	2.4977
		ER	1.4042	0.2298	0.3527	10.211	3.9308	0.7180	0.1280	0.1056
5	θ	AV	0.4253	0.4429	0.4099	0.3962	1.9835	1.5506	1.7532	1.5516
		ER	0.0656	0.0078	0.0089	0.0402	1.0335	1.8110	0.2419	1.8080
	λ	AV	1.8871	2.2850	1.6476	1.4811	3.9825	2.4977	3.1061	2.4982
		ER	1.1828	0.1764	0.2556	3.7556	3.9308	0.1056	0.1282	0.1058
10	θ	AV	0.1615	0.1642	0.1591	0.1568	1.9934	2.2067	1.7566	1.5536
		ER	0.0102	0.0012	0.0013	0.0056	8e-05	0.0056	0.0077	0.1163
	λ	AV	1.9151	2.2978	1.6824	1.5192	3.9932	4.0002	3.1087	2.4910
		ER	1.1461	0.1690	0.2439	3.1850	0.9866	0.1488	0.0015	0.1493

Table 5. MLEs and Bayesian parameter estimates derived from Prior I for $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{1}, \mathbf{1})$.

r			MLE	Bayes estimates using MCMC method				Bayes estimates using T–K			
				SEL	Linex			SEL	Linex		
					$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$		$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$
2	θ	AV	1.0640	1.0956	1.2805	0.9775	0.8929	1.8359	1.8081	1.48929	1.2136
		ER	0.4780	0.5725	0.0711	0.1046	0.8485	0.7173	0.1001	0.0274	0.0218
	λ	AV	1.7790	1.1397	1.3984	0.9955	0.8970	1.9051	1.8713	1.5289	1.2451
		ER	0.8780	0.7250	0.0938	0.1511	2.0399	0.8256	0.1137	0.0322	0.0277
3	θ	AV	1.0810	0.7150	0.8876	0.6506	0.6059	1.8832	1.8728	1.5055	1.2254
		ER	0.4760	0.3377	0.0395	0.0936	3.8120	0.7904	0.1134	0.0296	0.0238
	λ	AV	1.7230	1.1615	1.3974	1.0141	0.9107	1.9345	1.9172	1.5387	1.2476
		ER	0.8330	0.7235	0.0933	0.1375	1.2188	0.8769	0.1250	0.0333	0.0283
4	θ	AV	1.0800	0.4618	0.4969	0.4342	0.4117	1.9099	1.9089	1.5176	1.2315
		ER	0.4730	0.1223	0.0142	0.0180	0.0914	0.8346	0.1332	0.0309	0.0249
	λ	AV	1.6940	1.0677	1.2741	0.9359	0.8430	1.9504	1.9420	1.5434	1.2486
		ER	0.8070	0.6421	0.0810	0.01183	0.9723	0.9056	0.1317	0.3380	0.0285
5	θ	AV	1.0710	0.3176	0.3361	0.3027	0.2903	1.9263	1.9304	1.5248	1.2351
		ER	0.4760	0.0657	0.0076	0.0094	0.0459	0.8627	0.1287	0.0317	0.0257
	λ	AV	1.7010	1.0046	1.1926	0.8856	0.8009	1.9599	1.9565	1.5460	1.2491
		ER	0.8110	0.5784	0.0719	0.1064	0.9096	0.9230	0.1357	0.0341	0.0286
10	θ	AV	1.0210	0.1059	0.1076	0.1042	0.1027	1.9620	1.9735	1.5401	1.2426
		ER	0.4700	0.0067	0.0008	0.0009	0.0036	0.9261	0.1406	0.0334	0.0272
	λ	AV	1.7240	1.1412	1.3447	1.0099	0.9161	1.9800	1.9845	1.5512	1.2498
		ER	0.8220	0.6363	0.0805	0.1165	0.9485	0.9609	0.1439	0.0347	0.0287

Table 6. Bayesian estimates of the parameters under Prior II for $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{1}, \mathbf{1})$.

r			Bayes estimates using MCMC method				Bayes estimates using T–K			
			SEL	Linex			SEL	Linex		
				$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$		$\nu = -0.5$	$\nu = 0.5$	$\nu = 1$
2	θ	AV	0.9570	1.0349	0.8965	0.8474	1.8365	1.9030	1.6676	1.4992
		ER	0.2714	0.0327	0.0414	0.2316	0.7173	0.1290	0.0506	0.1070
	λ	AV	1.9250	2.3259	1.6874	1.5175	3.7825	3.8484	3.0152	2.4574
		ER	1.1914	0.0177	0.0217	0.1102	7.7822	1.7495	0.3732	0.6905
3	θ	AV	0.6950	0.7010	0.6251	0.5967	1.8828	1.9823	1.6956	1.5169
		ER	0.1496	0.0177	0.0217	0.1102	0.7894	0.1497	0.0544	0.1137
	λ	AV	1.8181	2.2077	1.5860	1.4260	3.8512	3.9140	3.0489	2.4738
		ER	1.1562	0.1698	0.2486	3.0420	8.1496	1.8450	0.3836	0.7030
4	θ	AV	0.4567	0.4777	0.4387	0.4230	1.9093	2.0309	1.7110	1.5264
		ER	0.0774	0.0092	0.0107	0.0496	0.8335	0.1638	0.0565	0.1174
	λ	AV	1.5688	1.8516	1.3844	1.2529	3.8885	3.9462	3.0660	2.4817
		ER	0.8917	0.1218	0.1708	1.5924	8.3558	1.8943	0.3890	0.7090
5	θ	AV	0.3308	0.3439	0.3160	0.3097	1.9257	2.0622	1.7202	1.5321
		ER	0.0482	0.0057	0.0066	0.0300	0.8616	0.1735	0.0578	0.1196
	λ	AV	1.4025	1.6630	1.2393	1.1238	3.9107	3.9636	3.0757	2.4859
		ER	0.7963	0.1057	0.1544	1.5822	8.4806	1.9217	0.3922	0.7122
10	θ	AV	0.1255	0.1273	0.1238	0.122	1.9616	2.1355	1.7339	1.5439
		ER	0.0070	0.0009	0.0009	0.0037	0.9622	0.1979	0.0607	0.1244
	λ	AV	1.3731	1.6394	1.2229	1.1159	3.9569	3.9920	3.0949	2.4939
		ER	0.7409	0.0984	0.1564	2.7061	8.7455	1.9681	0.3938	0.7184

Table 7. Point estimates and prediction intervals for $\boldsymbol{u_{r+1}}$

r	u_r	u_{r+1}	PI	MSE
		$(\theta, \lambda) = (2, 3)$		
2	0.01695691	0.0001620833	(0.0001417074,0.0001804214)	0.0004265947
3	0.01726967	0.0001618632	(0.00014183387,0.000180379)	0.0003912145
4	0.01735578	0.0001617438	(0.0001416832,0.0001804963)	0.0003714652
5	0.01738774	0.0001618371	(0.0001416945,0.0001803278)	0.0003568521
10	0.0175332	0.0001617736	(0.0001415956,0.0001803701)	0.0003323257
		$(\theta, \lambda) = (1, 1)$		
2	0.1022013	0.0001619977	(0.0001417149,0.0001804596)	0.01561456
3	0.1039733	0.0001618517	(0.0001416926,0.0001804655)	0.01437943
4	0.1036597	0.0001618021	(0.0001417286,0.0001804093)	0.001340301
5	0.1043488	0.0001619287	(0.0001416979,0.0001804013)	0.013030799
10	0.1050774	0.0001620208	(0.0001416760,0.0001803822)	0.0000121081

7.2. Data analysis

To illustrate the practical applicability of the proposed methods, we consider the following data sets representing the amount of rainfall (in inches) recorded from 1943 to 2006 at the Los Angeles Civic Center:8.08,7.35,11.47,21,27.36,8.11,24.35,12.46,12.40,31.01,9.09,11.57,17.94,4.42,16.49,9.24,3 7.25,13.19,3.21.

The corresponding lower record values for these data are presented in ***Table 8***.

Table 8: Record values based on rainfall data

i	1	2	3	4
l_i	8.08	7.35	4.42	3.21

Using the Kolmogorov-Smirnov test with MLEs $\hat{\theta} = 0.014$ and $\hat{\lambda} = 9.669$, we obtained test statistic 0.16 with p-value 0.67, indicating excellent fit.

Furthermore, diagnostic plots including the probability–probability (P–P) plot, the fitted versus empirical cumulative distribution functions, and the histogram with fitted probability density function are presented in ***Figure 1***.These plots clearly confirm the good fit of the model. In addition, the total time on test (TTT) plot shows a concave upward shape which suggests an increasing hazard rate. Since one of the fundamental properties of the Lindely–exponential distribution is the monotone increasing hazard function, the TTT plot further supports the suitability of this distribution for the given data set.

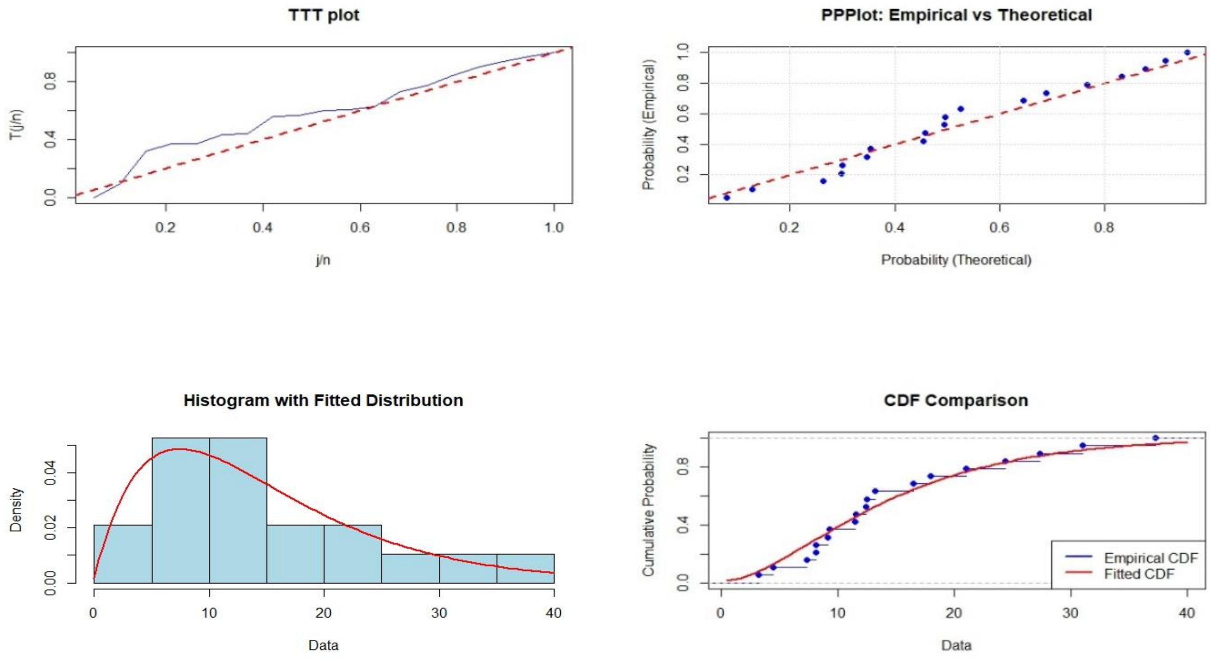


Figure 1. Total time on test (TTT) plot, probability–probability (P–P) plot, histogram of the data with the fitted probability density function and fitted versus empirical cumulative distribution functions

The results indicate that the maximum likelihood estimates (MLEs) obtained from the full dataset and the lower record values are comparable, despite the smaller number of observations in the latter. As shown in ***Table 9*** for the full dataset, the estimated parameters are $\hat{\theta} = 0.014$and $\hat{\lambda} = 9.669$, with a maximized log-likelihood of -66.71, indicating a good model fit. For the lower records with parameter constraints, the estimates are $\hat{\theta} = 0.012$and $\hat{\lambda} = 15$, with a maximized log-likelihood of -7.69.

Table 9.MLE estimates and log-likelihood values for the rainfall data

Estimation Method	$\hat{\theta}$	$\hat{\lambda}$	Log-likelihood (Full data)	Log-likelihood (Record data)
Full MLE	0.014	9.969	-66.71	-
Record MLE	0.012	15	-68.35	-7.69

The closeness of the θestimates demonstrates that the record-based method effectively preserves the main characteristics of the underlying distribution. Although $\hat{\lambda}$reaches its imposed upper limit, this reflects the limited information in the small number of records rather than a failure of the method. Moreover, ***Figure 2*** displays the histogram of the rainfall data along with the fitted probability density functions based on the full-data MLE and the record-based MLE. The near-overlapping curves clearly confirm the efficiency of the record-based estimation.

Overall, these results indicate that using lower record values can retain most of the essential information from the full dataset and provides an efficient approach for analyzing limited or extreme data points.

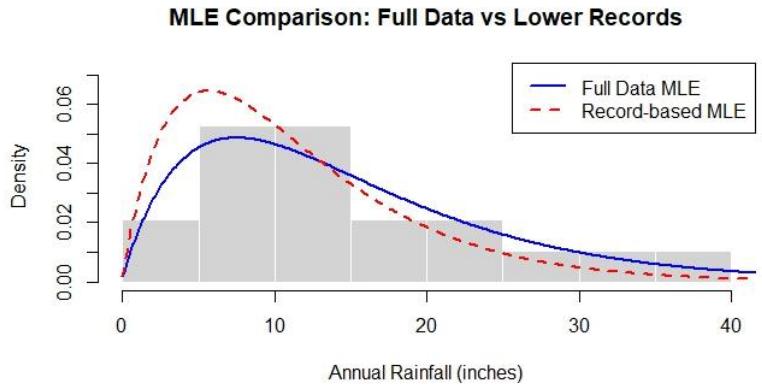
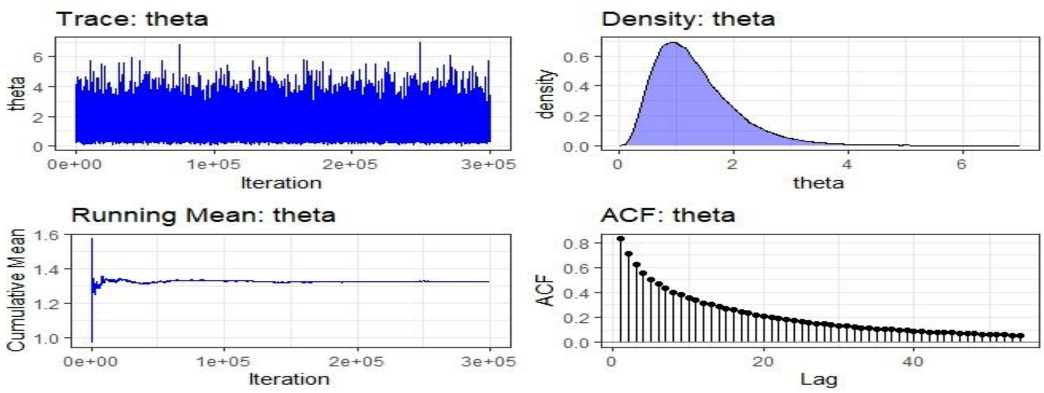


Figure 2. Histogram of rainfall data with fitted PDFs: Full MLE (blue solid line)

And record-based MLE (red dashed line).

For the Bayesian analysis, we consider two distinct sets of hyperparameter values: $(a, b) = (2,1), (c, d) = (2,1)$ (Prior I) and $(a, b) = (4,2), (c, d) = (4,1))$ (Prior II), in order to examine the sensitivity of the predictors and estimators to different informative priors. The approximate Bayes estimates of the parameters under the LINEX loss function ($\nu = 0.5,0.5,1$) and the squared error (SE) loss function are obtained using the methods of Tierney and Kadane as well as Gibbs sampling. The Bayesian estimates of the parameters are obtained using the MCMC method described in Section 3. Initially, 100,000 samples are generated from the posterior density given in equations ***Eq. (16)***. To reduce the influence of the initial samples, the first 30,000 observations are discarded. Then, to minimize autocorrelation among the generated samples, one observation is retained from every five iterations.

The plots of the simulated values of the parameters $\theta$and $\lambda$are presented in ***Figure 3*** and



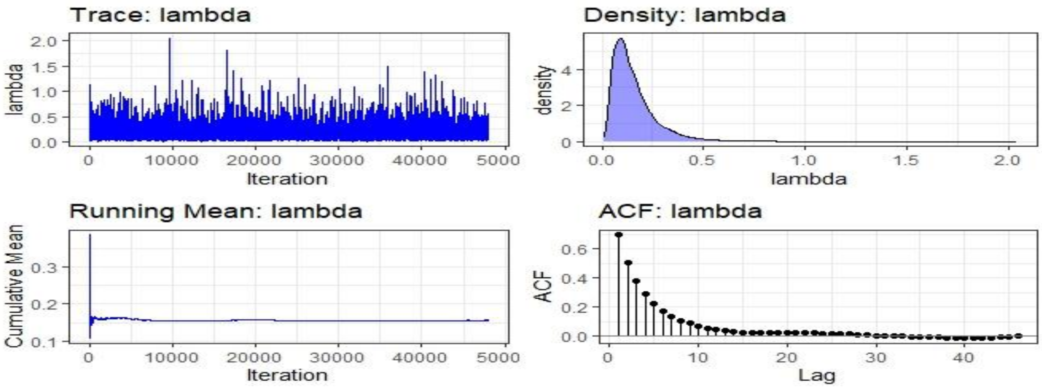


Figure 4 showing the convergence of the Gibbs sampling algorithm.

The MCMC diagnostics presented in the

***Table** 10* indicate that the chains for both parameters θ and λ under Prior I and Prior II have converged successfully. The Potential Scale Reduction Factor (M psrf) values are all very close to 1, confirming that the multiple chains have mixed well and reached the stationary posterior distribution. The acceptance rates are within reasonable ranges, suggesting that the proposal standard deviation (proposal-sd = 1) is appropriately tuned to allow efficient exploration of the posterior space.

The posterior estimates (P.est) for both parameters are stable and consistent across the two priors, and the upper bounds of the confidence intervals indicate concentrated posterior distributions. Although the acceptance rate for θ under Prior II is slightly lower than that under Prior I, it remains sufficient for effective sampling. Overall, these diagnostics confirm the reliability and robustness of the Bayesian estimates obtained via MCMC under different prior specifications.

***Table** 11* and

***Table** 12* present the Bayesian estimates of the parameters θ and λ of the Lindley–Exponential distribution based on lower record values. The estimations were obtained under two loss functions—the symmetric squared error loss (SEL) and the asymmetric LINEX loss—with various values of the asymmetry parameter ν, using both the Tierney–Kadane (T–K) approximation and the MCMC simulation method. The purpose is to compare the performance of these two Bayesian approaches and to evaluate the sensitivity of the estimates to the choice of loss function and prior distribution.

A comparison between the two tables indicates that the choice of prior has a notable influence on the posterior estimates. Under the informative prior (Prior II), posterior variances and expected risks (ER) are smaller, implying that informative priors lead to more precise and stable Bayesian estimates. Conversely, the vague prior (Prior I) yields wider HPD intervals and higher posterior variability, reflecting greater uncertainty in the inference. This pattern is fully consistent with Bayesian theory, since when data are limited, prior information significantly shapes the posterior distribution.

For both priors, the MCMC estimates of θ are generally larger than those obtained via the T–K approach. This difference arises because MCMC explores the full posterior distribution, while the T–K approximation relies on a local expansion around the posterior mode (MAP). Furthermore, as the asymmetry parameter ν in the LINEX loss function changes, the estimates shift systematically: for $\nu < 0$, estimates tend to be larger, whereas for $\nu > 0$, they become smaller compared with those under SEL. This reflects the directional sensitivity of the LINEX loss function which penalizes positive and negative estimation errors asymmetrically. In most cases, the expected risk (ER) under LINEX is smaller than under SEL, indicating that the LINEX loss may provide more efficient estimates when asymmetry in decision-making is relevant. For the parameter λ, both methods yield close and stable estimates. In most scenarios, MCMC produces slightly smaller posterior means than the T–K approximation, though the differences are minor and both results show good agreement. A noticeable reduction in ER under Prior II further confirms that informative priors reduce uncertainty and improve estimation accuracy. Moreover, the HPD intervals obtained under the second prior are shorter, suggesting greater posterior concentration around the true parameter values.

Table 10.Gelman–Rubin convergence diagnostics $(\hat{\boldsymbol{R}})$ for parameters **θ**and ***λ*** based on lower record values, under Prior I,II.

	Prior I					*Prior II*				
	P.est	Upper CI	M psrf	Acceptance rate	proposal_sd	P.est	Upper CI	M psrf	Acceptance rate	proposal_sd
θ	1.00	1.00	1.0005	0.5822	1	1.00	1.00	1.0003	0.4573	1
λ	1.00	1.00		0.2809	1	1.00	1.00		0.3426	1

Table 11. Bayes estimates of the parameters based on Prior I

		Bayes estimates using MCMC				Bayes estimates using T–K				HPD
		SEL	Linex			SEL	Linex			
			$v=-0.5$	$v=0.5$	$v=1$		$v=-0.5$	$v=0.5$	$v=1$	
θ	AV	2.1764	2.7335	1.8661	1.1694	0.1828	0.1885	0.1714	0.1657	(0.3220,4.6640)
	ER	1.5780	0.2785	0.1552	0.5171	0.0227	0.0028	0.0114	0.0259	
λ	AV	0.1345	0.1376	0.1317	0.1292	0.8034	0.9433	0.5354	0.3994	(0.0137,0.3380)
	ER	0.0116	0.0015	0.0014	0.0053	0.5439	0.0704	0.3125	0.8439	

Table 12. Bayes estimates of the parameters based on Prior II

		Bayes estimates using MCMC				Bayes estimates using T–K				HPD
		SEL	Linex			SEL	Linex			
			$v=-0.5$	$v=0.5$	$v=1$		$v=-0.5$	$v=0.5$	$v=1$	
θ	AV	1.3235	1.4612	1.2196	1.1372	0.5705	0.6196	0.4724	0.4232	(0.2598,2.6875)
	ER	0.4721	0.0688	0.0519	0.1862	0.1963	0.0248	0.1031	0.2471	
λ	AV	0.1562	0.1596	0.1531	0.1503	0.3460	0.3694	0.2992	0.2775	(0.0159,0.3732)
	ER	0.0129	0.0017	0.0015	0.0059	0.0937	0.0118	0.0480	0.1112	

predictive maximum likelihood estimator (PMLE) based on the likelihood function ***Eq. (29)*** the parameters were estimated as $\hat{\theta} = 0.0117$and$\hat{\lambda} = 29.506$.

Additionally, the values of the next five lower records for this data set are presented in ***Table 13***.The results indicate a gradual decrease in record values, with new lower records occurring at reasonable and consistent intervals.

Table 13. Predicted lower records based on PMLE

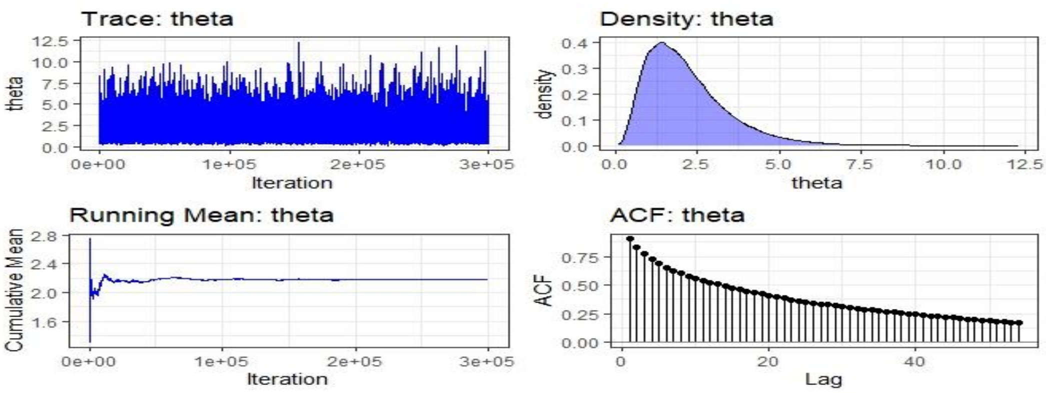
Record number	Predicted Lower Record
5	3.0061
6	2.9006
7	2.0701
8	2.0033
9	1.6767

Bayesian prediction based on the likelihood function ***Eq. (30)***, the values of the next five lower records for this data set, are presented in ***Table 14***.

Using MCMC with three independent chains, five future lower records were predicted for the Lindley-Exponential distribution. The acceptance rates of the chains were approximately 0.58 for θ and 0.32 for λ and the Gelman-Rubin diagnostic ($\hat{R}$) was close to 1, confirming the convergence of the chains and the reliability of the posterior samples.Predictions were performed using the posterior mean (Squared Error, SE) and the LINEX loss function ($v = -0.5, 0.5, 1$). The results indicate a decreasing trend in future lower records, and the differences between SE and LINEX predictions reflect the asymmetry of the predictive distribution and the sensitivity of the results to the choice of loss function.The predicted values show good agreement with the observed data, validating the Lindley-Exponential model and its ability to accurately capture the behavior of lower record values.

Table 14. Bayesian predicted values of the next five lower records.

Record number	SEL	Linex		
		$v = -0.5$	$v = 0.5$	$v = 1$
5	2.41	2.46	2.35	2.30
6	1.81	1.84	1.77	1.74
7	1.35	1.37	1.34	1.32
8	1.01	1.02	1.01	1.00
9	0.76	0.77	0.76	0.75



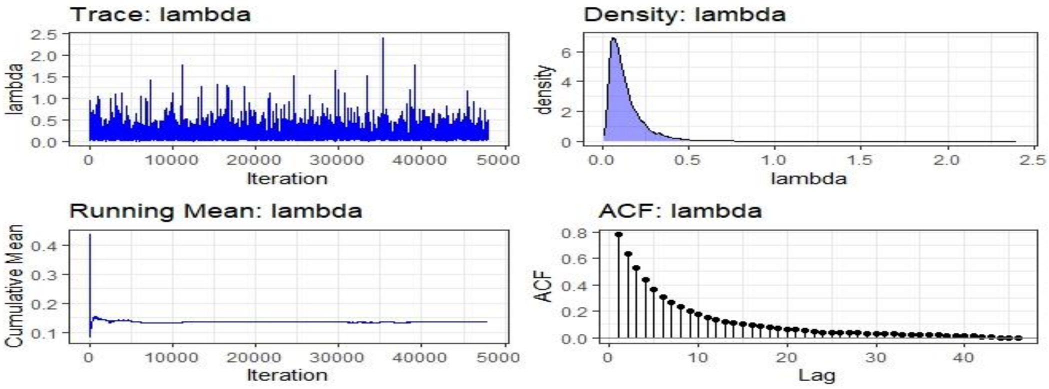


Figure 3. MCMC Convergence Plots for $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ under Prior I.

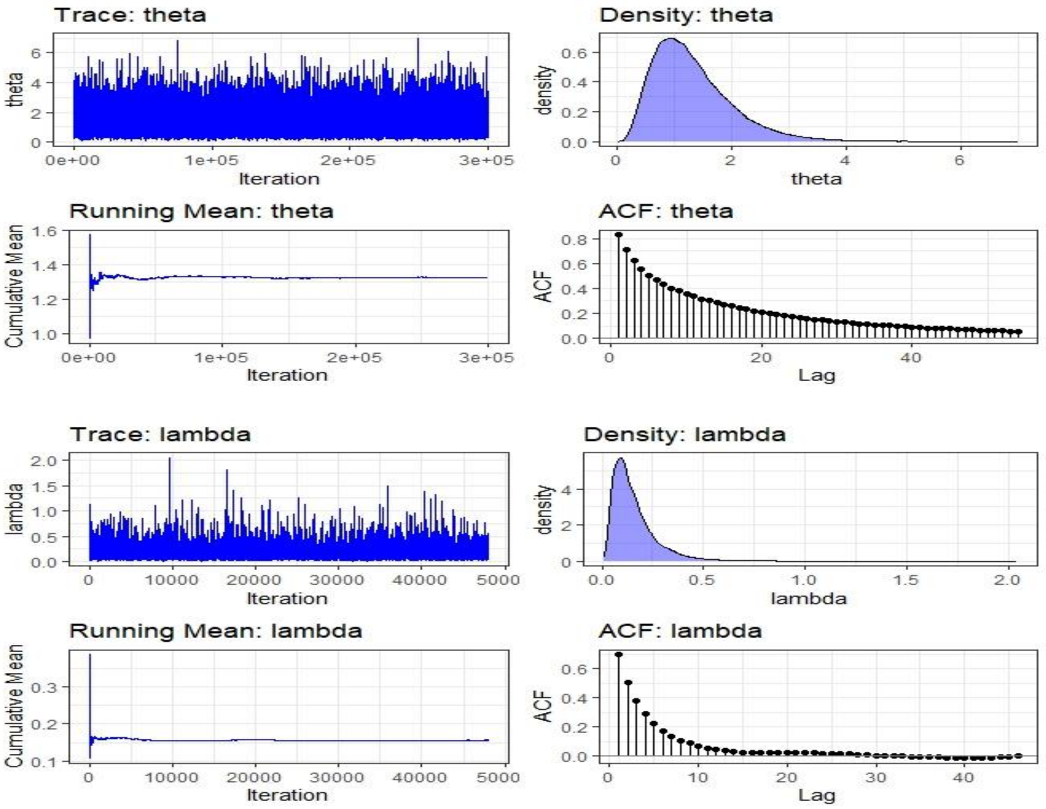


Figure 4. MCMC Convergence Plots for $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ under Prior II.

8. Conclusion

This study has thoroughly investigated statistical inference for the Lindley-Exponential (LE) distribution using lower record values. The key distributional properties of the model were derived, and both classical (frequentist) and Bayesian approaches were developed for parameter estimation and the prediction of future record values.

The main findings and contributions of this research can be summarized as follows:

1. **Model Properties and Record Theory:** We derived essential characteristics of the LE distribution, including the probability density function, cumulative distribution function, and

hazard rate. Furthermore, we established the distributional properties of lower record values from the LE model, such as their density and moments.

2. **Parameter Estimation:** Maximum likelihood estimators (MLEs) were derived, and asymptotic confidence intervals were constructed. For Bayesian inference, we employed independent gamma priors and obtained estimates under both squared error and LINEX loss functions. The computational challenges were addressed using the Tierney and Kadane approximation and a Metropolis-Hastings algorithm within a Gibbs sampling framework.
3. **Prediction of Future Records:** The study developed methods for predicting future lower record values. Both a maximum likelihood predictive approach and a full Bayesian predictive distribution, complete with point predictors and prediction intervals, were successfully implemented.
4. **Simulation Results:** Extensive Monte Carlo simulations demonstrated that the MLEs converge to the true parameter values as the sample size increases. The Bayesian estimators, particularly under the squared error loss, often achieved lower expected risks. The accuracy of predictors improved with the number of observed records and the Bayesian performance showed sensitivity to the choice of prior distributions and loss functions.
5. **Practical Utility and Future Research:** The methodologies developed here are highly applicable in fields like reliability analysis, meteorology and economics, where record value data is prevalent. For future research, these methods can be extended to analyze upper records, k-th records, or hybrid records. Other promising directions include incorporating censored data, developing regression models based on records and applying the framework to real-world datasets to further validate its practical utility.

In summary, this work provides a robust statistical toolkit for analyzing and predicting record-breaking data using the flexible Lindley-Exponential distribution, with both classical and Bayesian paradigms offering valuable insights for practitioners and researchers.

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Code Availability:

The custom R code that supports the findings of this study is not publicly available due to its length but can be obtained from the corresponding author upon direct request.

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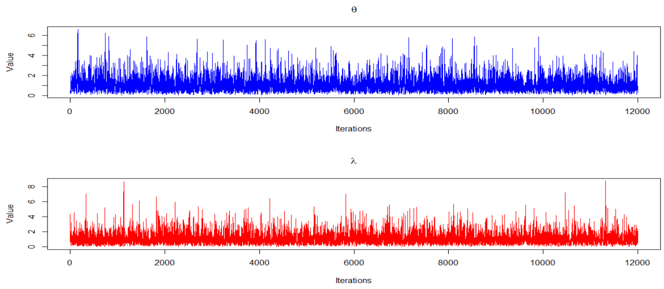
Appendix A:



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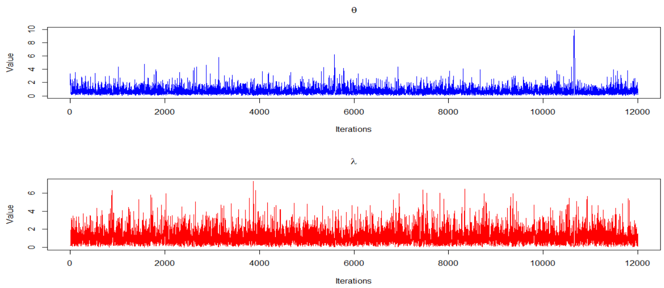

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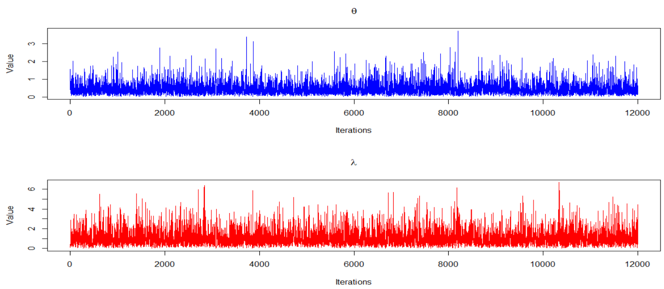

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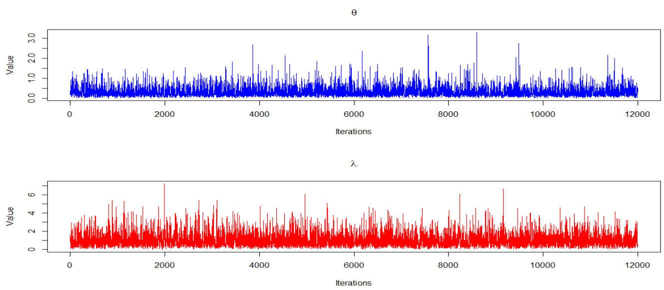

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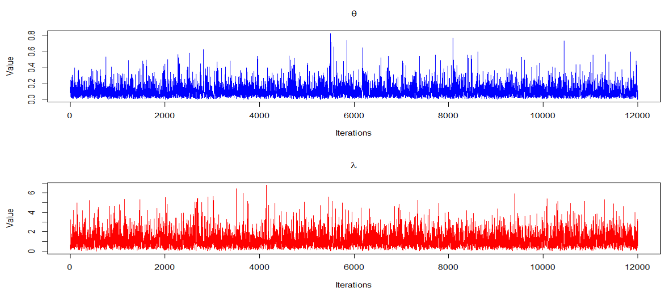

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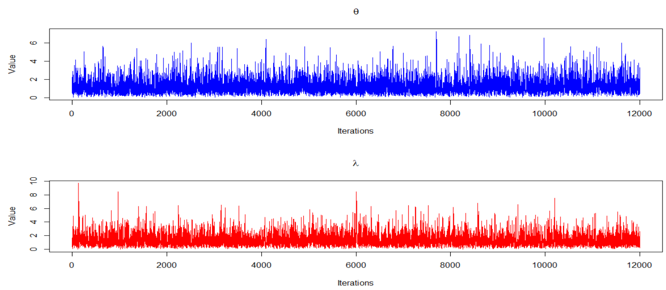
Figure 5. Trace Plots of θ and λ from MCMC Samples when $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{1}, \mathbf{1}), (\mathbf{PriorI})$ for different $\boldsymbol{r}$.



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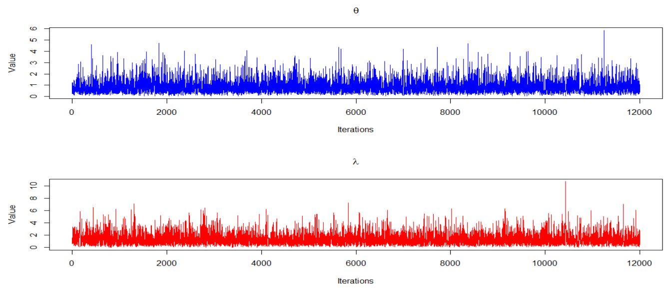

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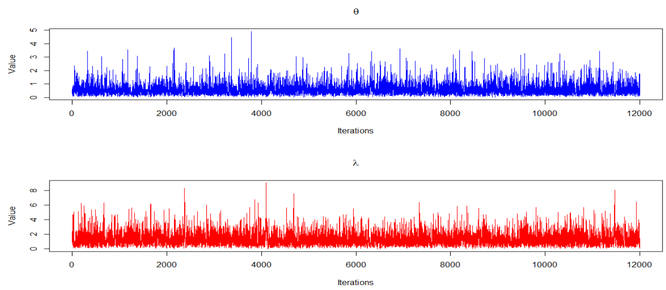

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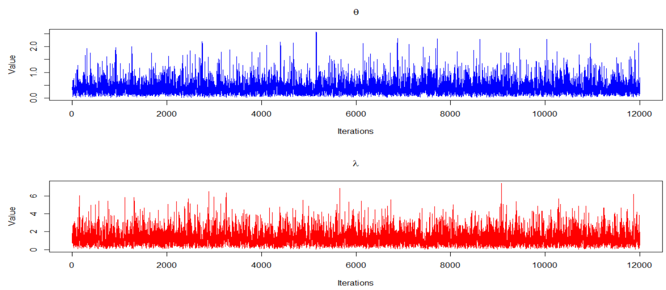

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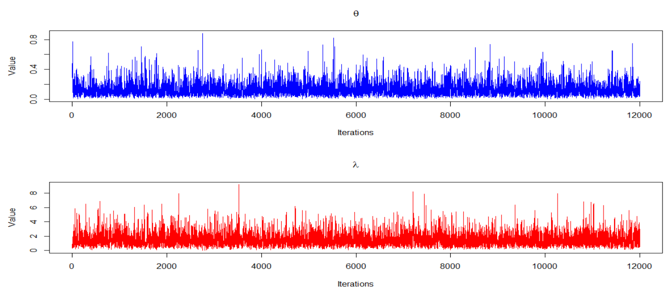
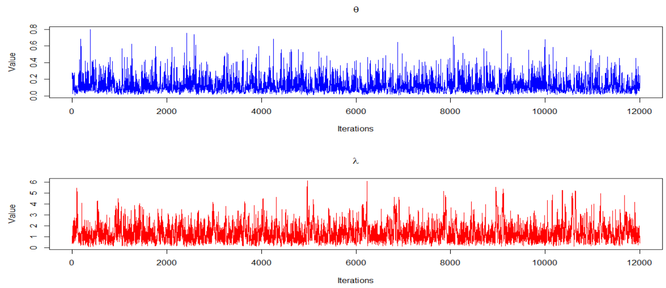


r = 10

Figure 6. Trace Plots of θ and λ from MCMC Samples when $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{2}, \mathbf{3})$, $(\textbf{PriorI})$ for different $\boldsymbol{r}$.



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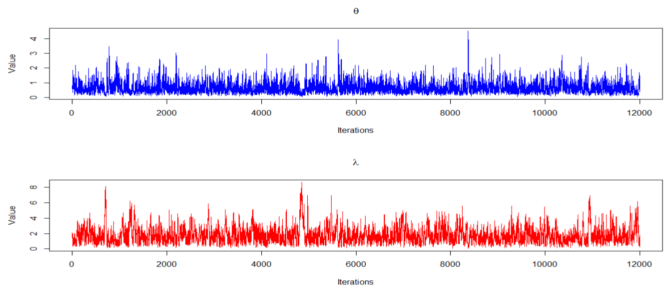

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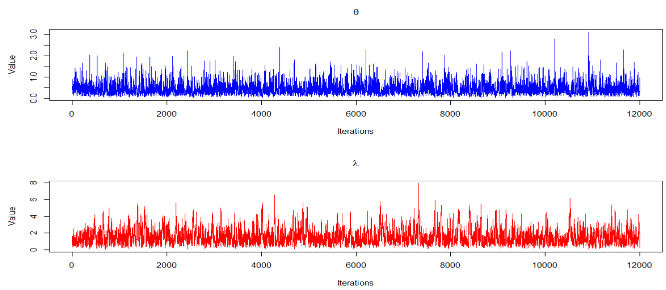

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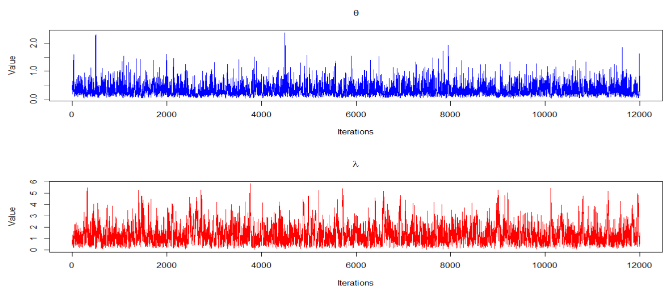

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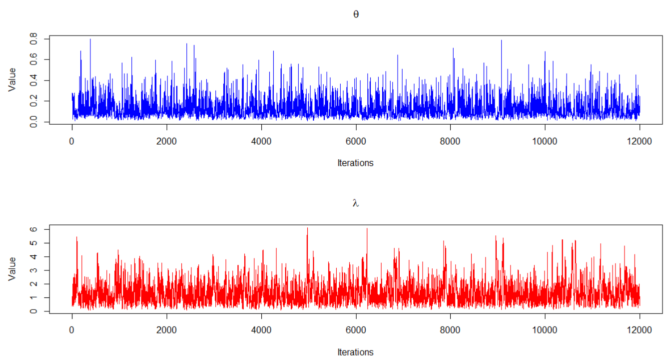
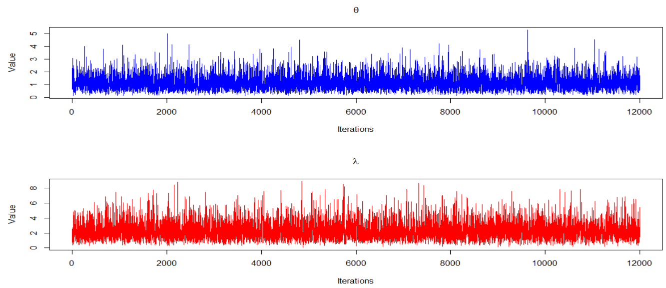


r = 10

Figure 7. Trace Plots of θ and λ from MCMC Samples when $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{1}, \mathbf{1})$, $(\mathbf{PriorII})$ for different $\boldsymbol{r}$.



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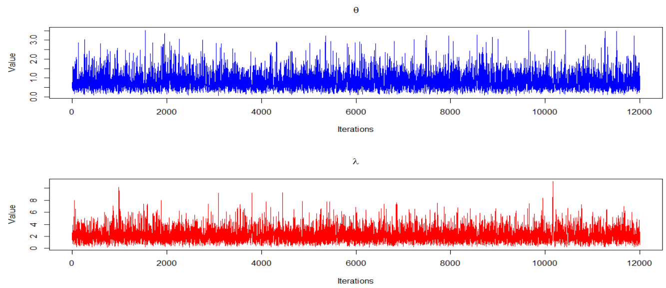

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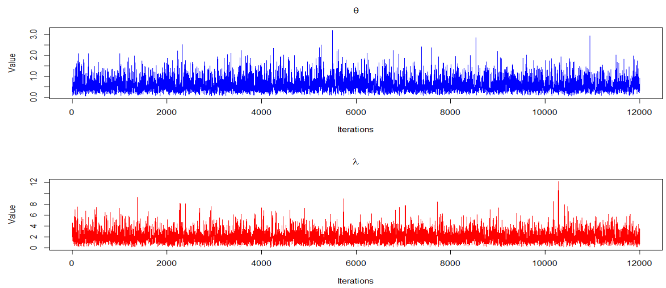

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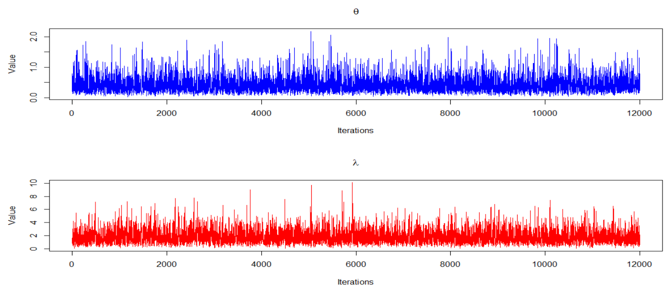

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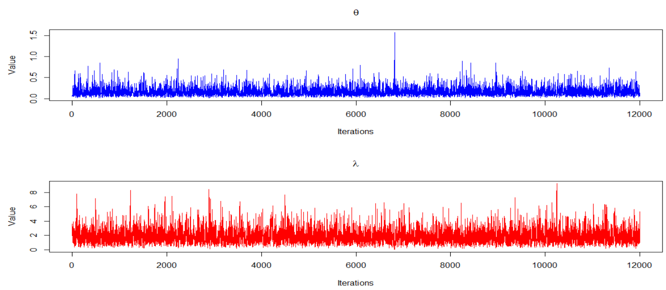

r = 10

Figure 8. Trace Plots of θ and λ from MCMC Samples when $(\boldsymbol{\theta}, \boldsymbol{\lambda}) = (\mathbf{2}, \mathbf{3}), (\mathbf{PriorII})$ for different $\boldsymbol{r}$.

Table 15. Gelman–Rubin convergence diagnostics $(\hat{\boldsymbol{R}})$ for parameters $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ based on lower record values, at different record sizes $(\boldsymbol{r})$, under Prior I.

r	parameter	$(\theta,\lambda)=(2,3)$			Acceptance rate	proposal_sd	$(\theta,\lambda)=(1,1)$			Acceptance rate	proposal_sd
		P.est	Upper CI	M psrf			P.est	Upper CI	M psrf		
2	$\boldsymbol{\theta}$	1.01	1.02	1.00	0.5421	1	1.00	1.00	1.00	0.4822	1
	$\boldsymbol{\lambda}$	1.00	1.01		0.5878	1	1.00	1.00		0.5122	1
3	$\boldsymbol{\theta}$	1.00	1.00	1.00	0.4164	1	1.00	1.03	1.00	0.3543	1
	$\boldsymbol{\lambda}$	1.01	1.02		0.5726	1	1.00	1.01		0.5089	1
4	$\boldsymbol{\theta}$	1.00	1.00	1.00	0.3024	1	1.00	1.00	1.00	0.4238	0.50
	$\boldsymbol{\lambda}$	1.00	1.00		0.5603	1	1.00	1.00		0.4770	1
5	$\boldsymbol{\theta}$	1.00	1.01	1.00	0.4510	0.4	0.40	1.00	1.00	0.3198	0.50
	$\boldsymbol{\lambda}$	1.00	1.01		0.5389	1	1.00	1.00		0.4505	1
10	$\boldsymbol{\theta}$	1.00	1.00	1.00	0.5318	0.1	1.00	1.01	1.00	0.4481	0.1
	$\boldsymbol{\lambda}$	1.00	1.00		0.5567	1	1.00	1.00		0.4631	1

Table 16. Gelman–Rubin convergence diagnostics $(\hat{\boldsymbol{R}})$ for parameters $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ based on lower record values, at different record sizes $(\boldsymbol{r})$, under Prior II.

r	parameter	$(\theta,\lambda)=(2,3)$			Acceptance rate	proposal_sd	$(\theta,\lambda)=(1,1)$			Acceptance rate	proposal_sd
		P.est	Upper CI	M psrf			P.est	Upper CI	M psrf		
2	θ	1.00	1.00	1.00	0.4578	1	1.00	1.00	1.00	0.3964	1
	λ	1.00	1.00		0.6875	1	1.00	1.00		0.6186	1
3	θ	1.00	1.00	1.00	0.3508	1	1.00	1.01	1.00	0.3228	0.90
	λ	1.00	1.00		0.6607	1	1.00	1.02		0.5862	1
4	θ	1.01	1.03	1.00	0.3167	1	1.00	1.00	1.00	0.2594	0.80
	λ	1.01	1.02		0.6232	1	1.00	1.00		0.5413	1
5	θ	1.00	1.00	1.00	0.3549	0.4	0.40	1.00	1.00	0.2864	0.50
	λ	1.00	1.00		0.5971	1	1.00	1.00		0.5018	1
10	θ	1.00	1.00	1.00	0.3356	0.1	1.00	1.01	1.00	0.4452	0.1
	λ	1.00	1.00		0.5761	1	1.01	1.01		0.4746	1