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A Unified Analytical Method for Generating Diverse Solutions of the Unstable Schrödinger Equation

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ABSTRACT

This research introduces an innovative implementation of the enhanced Jacobi Elliptic function framework to investigate the unstable nonlinear Schrödinger equation. This sophisticated computational approach demonstrates remarkable efficacy in generating comprehensive solution families, offering substantial practical utility across mathematical physics applications. The methodology facilitates the derivation of multiple distinct solution categories with clearly characterized properties. Computational visualization techniques effectively elucidate the dynamic behavioral patterns exhibited by the obtained solutions.

1. Introduction

Contemporary modeling challenges across engineering disciplines, empirical sciences, and socio-economic systems frequently necessitate mathematical formulations whose resolution depends on determining unknown functions satisfying derivative-based relationships. The historical development of partial differential equations traces back to foundational contributions from eighteenth-century mathematical pioneers including Euler, Lagrange, D'Alembert, and Laplace, who established analytical frameworks for physical system characterization. The comprehensive domain of equation theory encompassing both ordinary and partial differential equations—represents a vital branch of mathematical analysis maintaining profound interconnections with physical principles. Fundamental physical concepts spanning vibrating string dynamics, gravitational interactions, fluid mechanics, thermal transport phenomena, and electromagnetic field theory have consistently motivated the development of diverse differential equation formulations. The solutions to these equations provide crucial insights into underlying physical mechanisms. Conventional derivative-based models often prove inadequate for capturing numerous complex

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physical phenomena, whereas fractional derivative formulations frequently yield enhanced observational alignment and improved predictive capability. Beyond the conventional velocity interpretation associated with ordinary derivatives in mechanical physics, fractional derivatives introduce additional dimensionality to system characterization, particularly through accelerated or decelerated motion descriptors.

Recent methodological advances have produced numerous innovative approaches for obtaining exact solutions to nonlinear partial differential equations. Among these, the refined Jacobi Elliptic function technique [1-2] stands out as particularly noteworthy. This method represents a powerful analytical tool for determining precise wave solutions, establishing itself as one of the most straightforward yet potent algebraic strategies for addressing nonlinear partial differential systems [3-7].

The distinctive advantage of this methodology over conventional approaches lies in its ability to circumvent computationally intensive procedures while delivering highly accurate, explicit solitary wave solutions with remarkable precision. In contemporary research practice, this technique has demonstrated significant utility across various scientific and engineering domains, successfully generating exact solutions including soliton structures, alternating patterns, and exponential forms for Schrödinger-type equations, Eckhaus systems, and numerous other nonlinear partial differential equations [8-18]. The current investigation employs the advanced enhanced Jacobi Elliptic function framework to construct traveling wave solutions for the unstable nonlinear Schrödinger equation:

$$i\frac{\partial v(x,t)}{\partial t}+\frac{\partial^2 v(x,t)}{\partial x^2}+\ 2\alpha\mid v(x,t)\mid^2\ v(x,t)-\gamma\frac{\partial^2 v(x,t)}{\partial x\partial t}=\ 0. \tag{1}$$

In the following, we present the method in question in the second part, and in the third part, we describe the application of the method along with the graphical analysis of the behavior of the answers. And in the final part, we present the results of the discussion.

2. Enhanced Jacobi Elliptic Function Framework

The core procedure of this analytical framework is outlined in the following sequence. The process begins with a nonlinear partial differential equation of the general form:

$$\mathcal{L}(\psi,\psi_x,\psi_t,\psi_{xx},\ldots)=0. \tag{2}$$

Application of the wave variable transformation $\psi=\varphi(\xi)$ and $\xi=\sigma x-lt$ condenses this PDE into an ordinary differential equation:

$$\mathcal{L}(\varphi,\sigma\varphi',-l\varphi',\ldots)=0, \tag{3}$$

The parameters σ and l are to be determined subsequently.

Stage 1: The solution for Equation (3) is postulated as a finite series:

$$\varphi(\xi) = g_0 + \sum_{j=1}^{M}\left[g_j\left(\frac{z(\xi)}{1+z^2(\xi)}\right) + f_j\left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)\right]\left(\frac{z(\xi)}{1+z^2(\xi)}\right)^{j-1} \quad (4)$$

where the function $z(\xi)$ is governed by the first-order differential relation:

$$z(\xi) = \sqrt{a + bz^2(\xi) + cz^4(\xi)} \quad (5)$$

here $g_0, g_j, f_j, a, b, c\,(j = 1, 2, ..., N)$ are constants that need to be evaluated.

Stage 2: The positive integer M is ascertained by applying the homogeneity balance principle to the nonlinear components and the highest-order derivative present in Equation (3). Substituting the ansatz (4) in conjunction with the auxiliary equation (5) into the ODE (3) yields an expression composed of powers of $z(\xi)$.

Stage 3: Setting the coefficients of all terms z^j from the resulting polynomial to zero generates a set of algebraic conditions. This system is then solved computationally using symbolic software such as Mathematica or Maple to ascertain the values of the unknown constants such as $g_0, g_j, f_j\,(j = 1, 2, ..., N)$.

Step (2-4): The general solutions of ***Eq. (5)*** can be obtained as follows:

- If $a = 1, b = -1 - m^2, c = m^2$ the general solution of (5) is $sn(\xi, m)$.
- If $a = 1 - m^2, b = 2m^2 - 1, c = -m^2$ the general solution of (5) is $cn(\xi, m)$.
- If $a = -m^2, b = 2m^2 - 1, c = 1 - m^2$ the general solution of (5) is $nc(\xi, m)$.
- If $a = \frac{1}{4}, b = \frac{1-2m^2}{2}, c = \frac{1}{4}$ the general solution of (5) is $ns(\xi, m) \pm cs(\xi, m)$.
- If $a = \frac{1-m^2}{4}, b = \frac{1+m^2}{2}, c = \frac{1-m^2}{4}$, the general solution of (5) is $nc(\xi, m) \pm sc(\xi, m)$

By substituting the identified parameters $g_0, g_j, f_j\,(j = 1, 2, ..., N), c$ and the general solutions of ***Eq. (5)*** into ***Eq. (4)***, one can obtain several exact travelling-wave solutions of the differential equation (2).

3. Implementations

A novel wave variable is introduced as follows:

$$u(x,t) = U(\xi)e^{i\phi(x,t)}, \xi = kx - \omega t, \phi(x,t) = \sigma x + \mu t, \quad (6)$$

The solution for ***Eq. (1)*** is conjectured to possess a specific form, with k, ω, and μ representing predefined constants. Upon applying the variable substitution from (6) to ***Eq. (1)***, the resulting equation is separated by setting the coefficients of its real and imaginary components individually to zero. Subsequently, by defining the relationship $\omega = \dfrac{k\left(\gamma\mu - 2\sigma\right)}{\gamma\mu - 1}$,, the following nonlinear ordinary differential equation is obtained:

$$2\alpha U^{3}\left(\xi\right) + \left(\gamma\mu\sigma - \mu - \sigma^{2}\right)U\left(\xi\right) + \kappa\left(\gamma\mu + \kappa\right)\frac{d^{2}U}{d\xi^{2}} = 0. \tag{7}$$

By applying the HBM between and we have $M = 1$so we have

$$U\left(\xi\right) = g_{0} + g_{1}\left(\frac{z\left(\xi\right)}{1 + z^{2}\left(\xi\right)}\right) + f_{1}\left(\frac{1 - z^{2}\left(\xi\right)}{1 + z^{2}\left(\xi\right)}\right), \tag{8}$$

By substituting the expression from (8) into equation (7) and setting the sum of the coefficients for each power of the function to zero, a set of algebraic relations is generated. This system of equations is then solved computationally using a symbolic algebra package such as Maple, leading to the determination of the unknown constants, which are presented as distinct cases below:

Family 3-1: If we consider $a = 1, b = -1 - m^{2}, c = m^{2}$ and substituting in algebraic equation we obtain following cases of solutions

Case 3-1-1:

$$g_{0} = 0, g_{1} = \frac{i\sqrt{3}k}{2}, f_{1} = \frac{(1 - \sqrt{k})^{(1/3)}}{2},$$
$$\sigma = \frac{1}{2}, \mu = \frac{k}{2}, \omega = \sqrt{k}, m = 0.$$

In this case $sn\left(\chi, 0\right)$ is $\sin\left(\chi\right)$. So, ***Eq. (1)*** admits the following solution (see ***Figure 1***):

$$u_{1}\left(x,t\right) = \frac{i\sqrt{3}k}{2}\left(\frac{\sin\left(kx - \sqrt{k}t\right)}{1 + \sin^{2}\left(kx - \sqrt{k}t\right)}\right)e^{i\left(\frac{1}{2}x + \frac{k}{2}t\right)} +$$
$$\frac{(1 - \sqrt{k})^{(1/3)}}{2}\left(\frac{1 - \sin^{2}\left(kx - \sqrt{k}t\right)}{1 + \sin^{2}\left(kx - \sqrt{k}t\right)}\right)e^{i\left(\frac{1}{2}x + \frac{k}{2}t\right)},$$

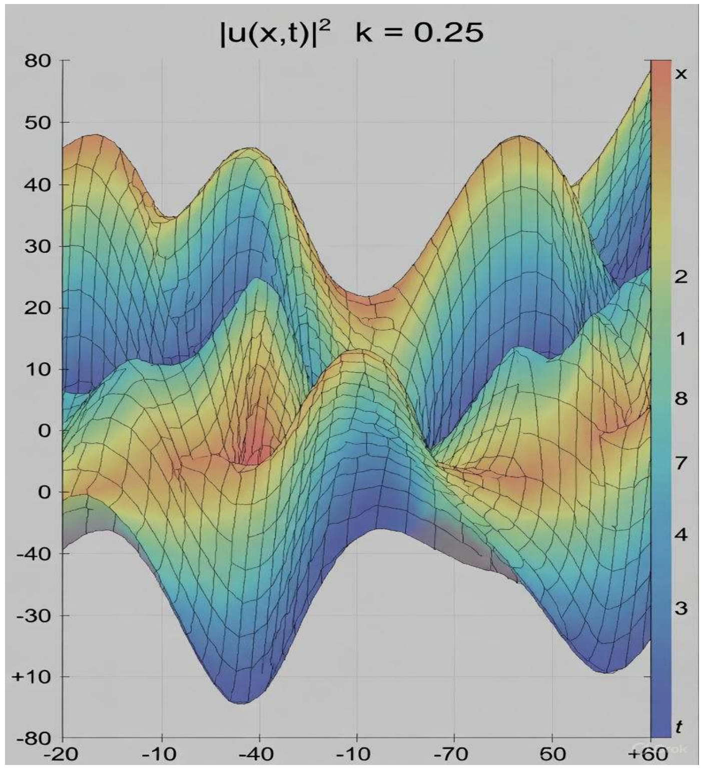

Figure 1. 3D-geraphical behavior of $u_1\left(x,t\right)$

Case 3-1-2:

$$g_0 = 0, g_1 = -\frac{i\sqrt{3}k}{2}, f_1 = -\frac{(1-\sqrt{k})^{(1/3)}}{2},$$
$$\sigma = \sqrt{\frac{1}{2}}, \mu = \frac{\sqrt{k}}{2}, \omega = \sqrt{k}, m = 1.$$

In this case $sn\left(\chi,1\right)$ is $\tanh\left(\chi\right)$. So, ***Eq. (1)*** admits the following solution (see ***Figure 2***):

$$u_2\left(x,t\right) = -\frac{i\sqrt{3}k}{2}\left(\frac{\sin\left(kx-\sqrt{k}t\right)}{1+\sin^2\left(kx-\sqrt{k}t\right)}\right)e^{i\left(\sqrt{\frac{1}{2}}x+\frac{\sqrt{k}}{2}t\right)} -$$
$$\frac{(1-\sqrt{k})^{(1/3)}}{2}\left(\frac{1-\sin^2\left(kx-\sqrt{k}t\right)}{1+\sin^2\left(kx-\sqrt{k}t\right)}\right)e^{i\left(\sqrt{\frac{1}{2}}x+\frac{\sqrt{k}}{2}t\right)},$$

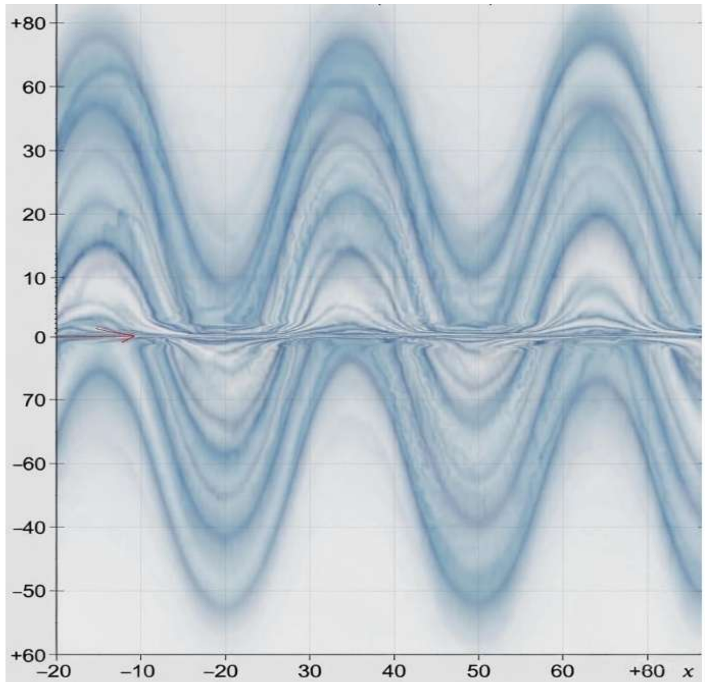

Figure 2. 3D-geraphical behavior of $u_1(x,t)$

Family 3-2: If we consider $a = 1 - m^2, b = 2m^2 - 1, c = -m^2$ and substituting in algebraic equation we obtain following cases of solutions

Case3-2-1:

$$g_0 = -\frac{i\sqrt{5}\left(k^2+1\right)}{3}, g_1 = 0, f_1 = \frac{(1+\sqrt{k})\sigma^{(1/3)}}{2},$$
$$\mu = \frac{1}{2}, \omega = \sqrt{k+1}, m = 0.$$

In this case $cn(\chi, 0)$ is $\cos(\chi)$. So, ***Eq. (1)*** admits the following solution (see ***Figure 3***):

$$u_3(x,t) = -\frac{i\sqrt{5}\left(k^2+1\right)}{3} + \frac{(1+\sqrt{k})\sigma^{(1/3)}}{2}\left(\frac{1-\sin^2\left(kx-\sqrt{k+1t}\right)}{1+\sin^2\left(kx-\sqrt{k+1t}\right)}\right)e^{i\left(\sqrt{\frac{1}{2}}x+\frac{1}{2}t\right)},$$

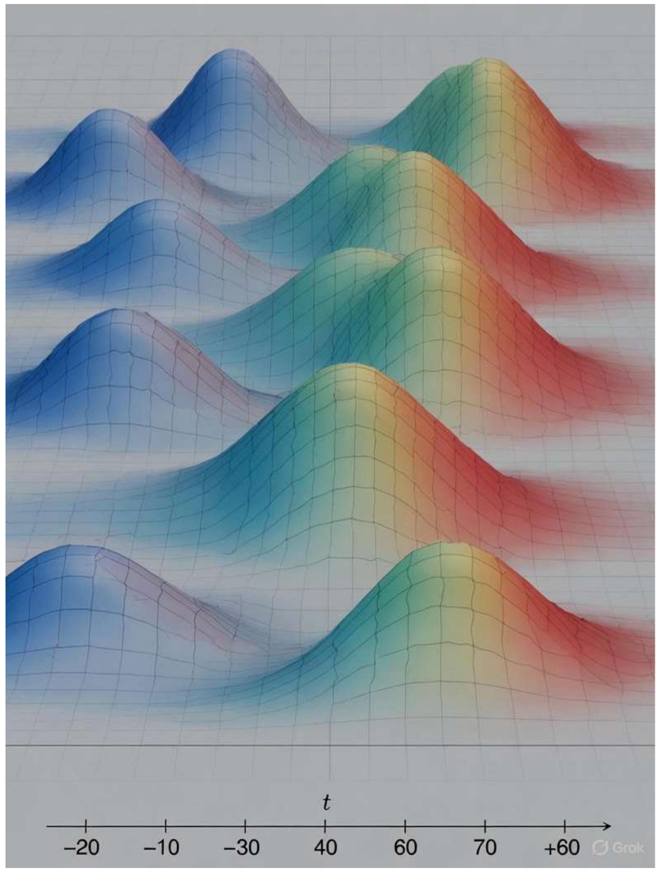

Figure 3. 3D-geraphical behavior of $u_3\left(x,t\right)$

Case3-2-2: we obtain

$$g_0=\frac{i\sqrt{5}\left(k^2-1\right)}{3},g_1=0,f_1=\frac{(1-\sqrt{k})^{(1/3)}}{2},$$
$$\sigma=-1,\mu=-\frac{1}{2},\omega=-\sqrt{k+1},m=1.$$

In this case $sn\left(\chi,1\right)$ is $\operatorname{sech}\left(\chi\right)$. So, ***Eq. (1)*** admits the following solution (see ***Figure 4***):

$$u_4\left(x,t\right)=\frac{i\sqrt{5}\left(k^2-1\right)}{3}+$$
$$\frac{(1-\sqrt{k})^{(1/3)}}{2}\left(\frac{1-\sin^2\left(kx+\sqrt{k+1}t\right)}{1+\sin^2\left(kx+\sqrt{k+1}t\right)}\right)e^{i\left(-x-\frac{1}{2}t\right)},$$

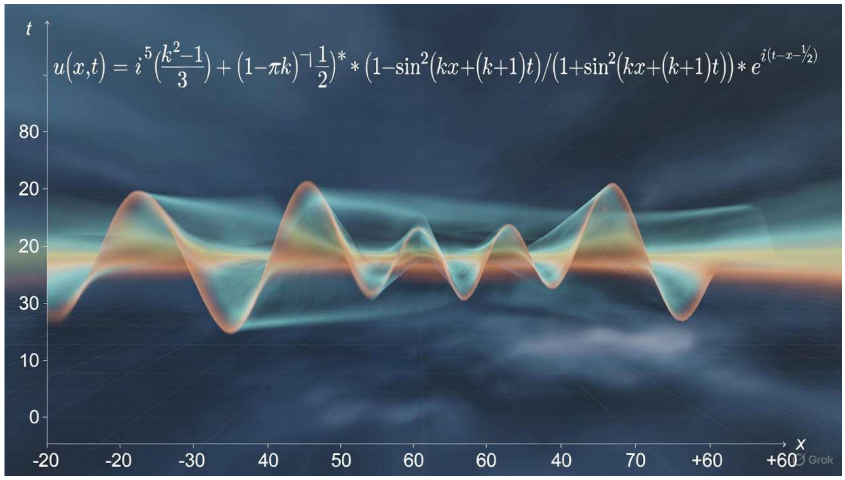

Figure 4. 3D-geraphical behavior of $u_4(x,t)$

Family 3-3: If we consider $a=-m^2, b=2m^2-1, c=1-m^2$ and substituting in algebraic equation we obtain following cases of solutions

Case 3-3-1:

$$g_0=\frac{1}{2}, g_1=\frac{k^2-1}{3}, f_1=\frac{(1-\sqrt{k})}{2},$$
$$\sigma=-\frac{1}{2}, \mu=-\frac{1}{2}, \omega=-\sqrt{k}, m=0.$$

In this case, $nc(\chi,0)$ is deduced to $\operatorname{sech}(\chi)$. So, ***Eq. (1)*** has the following solution:

$$u_5(x,t)=\frac{1}{2}+\frac{k^2-1}{3}\left(\frac{\sin\left(kx-\sqrt{k}t\right)}{1+\sin^2\left(kx-\sqrt{k}t\right)}\right)e^{i\left(-\frac{1}{2}x-\frac{1}{2}t\right)}+$$
$$\frac{(1-\sqrt{k})}{2}\left(\frac{1-\sin^2\left(kx+\sqrt{k}t\right)}{1+\sin^2\left(kx+\sqrt{k}t\right)}\right)e^{i\left(-\frac{1}{2}x-\frac{1}{2}t\right)},$$

Case3-3-2:

$$g_0=\frac{\sigma}{2}, g_1=\frac{k^2-1}{3}, f_1=0,$$
$$\mu=-\sqrt{\frac{1}{2}}, \omega=-\sqrt{k+1}, m=1.$$

In this case, $nc(\chi,1)$ is deduced to $\cosh(\chi)$. So, ***Eq. (1)*** has the following solution:

$$u_6(x,t)=\frac{\sigma}{2}+\frac{k^2-1}{3}\left(\frac{\sin\left(kx+\sqrt{k+1}t\right)}{1+\sin^2\left(kx+\sqrt{k+1}t\right)}\right)e^{i\left(-\frac{1}{2}x-\sqrt{\frac{1}{2}}t\right)},$$

Family 3-4: If we consider $a=\frac{1}{4}, b=\frac{1-2m^2}{2}, c=\frac{1}{4}$ and substituting in algebraic equation we obtain following cases of solutions:

$$g_0=\frac{3\sigma}{2}, g_1=\frac{\sqrt{k^2-1}}{3}, f_1=\frac{\sqrt{k}}{2},$$
$$\mu=-\sqrt{\frac{3}{2}}, \omega=-\sqrt{k^2+1}, m=1.$$

In this case, $ns(\chi,1)+cs(\chi,1)$ is deduced to $\coth(\chi)+\operatorname{csch}(\chi)$. So, ***Eq. (1)*** has the following solution:

$$u_7(x,t)=\frac{3\sigma}{2}+\frac{\sqrt{k^2-1}}{3}\left(\frac{\sin\left(kx+\sqrt{k^2+1t}\right)}{1+\sin^2\left(kx+\sqrt{k^2+1t}\right)}\right)e^{i\left(-\frac{1}{2}x+\sqrt{\frac{3}{2}}t\right)}+$$
$$\frac{\sqrt{k}}{2}\left(\frac{1-\sin^2\left(kx+\sqrt{k^2+1t}\right)}{1+\sin^2\left(kx+\sqrt{k^2+1t}\right)}\right)e^{i\left(-\frac{1}{2}x+\sqrt{\frac{3}{2}}t\right)},$$

Family 3-5: If we consider $a=\frac{1-m^2}{4}, b=\frac{1+m^2}{2}, c=\frac{1-m^2}{4}$, and substituting in algebraic equation we obtain following solution

$$g_0=\frac{\sqrt{3\sigma}}{2}, g_1=\frac{\sqrt{k^2+1}}{3}, f_1=-\frac{\sqrt{k}}{2},$$
$$\mu=\sqrt{\frac{3\sigma}{2}}, \omega=\sqrt{k^2+1}, m=1.$$

In this case $nc(\chi,1)+sc(\chi,1)$ is $\cosh(\chi)+\sinh(\chi)$. So, ***Eq. (1)*** admits the following solution (see ***Figure 5***):

$$u_8(x,t)=\frac{\sqrt{3\sigma}}{2}+\frac{\sqrt{k^2+1}}{3}\left(\frac{\sin\left(kx-\sqrt{k^2+1t}\right)}{1+\sin^2\left(kx-\sqrt{k^2+1t}\right)}\right)e^{i\left(-\frac{1}{2}x+\sqrt{\frac{3\sigma}{2}}t\right)}-$$
$$\frac{\sqrt{k}}{2}\left(\frac{1-\sin^2\left(kx-\sqrt{k^2+1t}\right)}{1+\sin^2\left(kx-\sqrt{k^2+1t}\right)}\right)e^{i\left(-\frac{1}{2}x+\sqrt{\frac{3\sigma}{2}}t\right)},$$

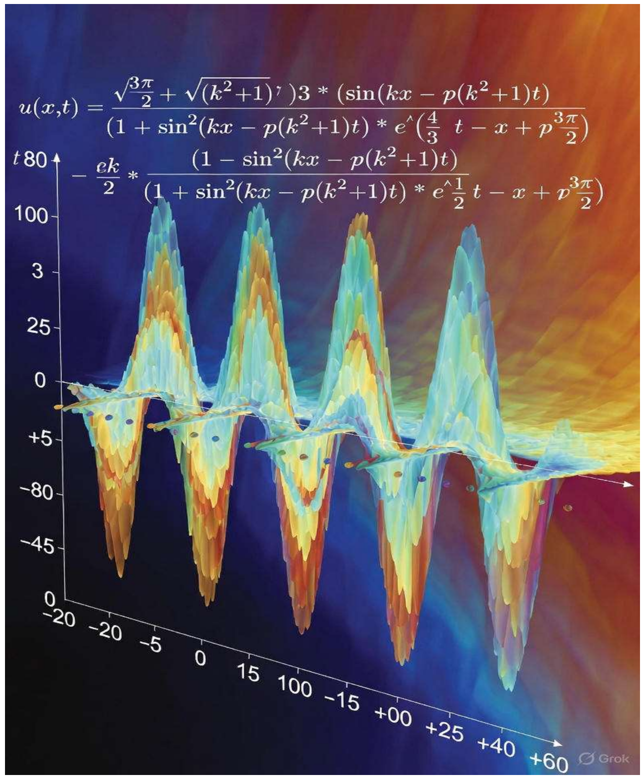

Figure 5. 3D-geraphical behavior of $u_8\left(x,t\right)$

4. Conclusions:

This investigation has demonstrated the successful implementation of the enhanced Jacobi Elliptic function methodology for deriving comprehensive solution families to the unstable Schrödinger equation. The computational framework and resultant solutions collectively highlight the methodological simplicity and remarkable solution diversity afforded by this approach. The technique exhibits substantial potential for application to numerous other equations previously addressed through alternative methods that may not yield comparably diverse solution structures.

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